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# Fractional quantum Hall effects as an example of fractal geometry in nature

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**Abstract.** A prescription is provided for constructing the Hall curve including both integral (I)- and fractional (F)-quantum Hall effects (QHE) that is based upon the iterative application of particular transformations simultaneously to the Hall resistance ($R_{xy}$) and magnetic field (B) axes of a template constructed from the elementary (integral quantum) Hall curve to filling factor $\nu = 1$. The construction shows that scaled copies of the elementary Hall curve reappear in various parts of the constructed curve upon increasing the magnification, resulting in FQHE sequences in higher Landau bands, and novel FQHE sequences between main sequence FQHE's in the lowest Landau band. The self similarity observed in the constructed Hall curve helps to draw a connection between FQHE's and the classical problem of an electron-in-a-periodic-potential-subjected-to-a-magnetic-field ('Hofstadter's butterfly'), and suggests that fractional quantum Hall effects constitute another manifestation of fractal geometry in nature–one that might also be viewed as a signature of transport in a Wigner crystal.

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Fractals are objects that appeal to the intuition by exhibiting self-similarity upon the transformation of the characteristic scales (scaling) associated with the object. A study of some simple fractals shows that this feature may be traced to the construction of the fractal by the repeated application of well specified operations upon an elementary unit or 'fractal generator'. The important role for the construction- and the pictorial representation- in the study of fractals has also been reinforced by the collection of fractals studied by Mandelbrot [1]. These points suggest that establishing fractal characteristics in physical phenomena involves both the identification of the elementary units, and the prescription for constructing the object, so that the origin of the self-similar fine structure that appears upon 'increasing the magnification' might be understood by appealing to the intuition [1].

The Hall effect is an old effect which has enjoyed a remarkable resurgence following the discovery of Hall resistance quantization, in a two dimensional electron system, at high magnetic fields (B) and low temperatures (T) [2–26]. The theoretical study of quantum Hall effects at integral- and (predominantly) odd denominator fractional- filling factors ($\nu$) has recently led to the examination of the half filled Landau band in the two-dimensional electron system (2DES), and suggestions of its equivalence to the zero magnetic field (B = 0) situation, according to Halperin [14, 15]. This appealing analogy has served as a catalyst for several recent experiments that have been carried out in order to confirm the Fermi surface at $\nu = 1/2$, and the novel quasiparticles that constitute the Fermi sea [16–26]. These developments have suggested a linkage between fractional quantum Hall effects occuring in the vicinity of filling factors $\nu = p/(2mp \pm 1)$ within the lowest Landau band, with m, p, = positive integers, with integral quantum Hall effects at $\nu = p$ and, in a sense, linked together the integral- and the fractional- aspects of the problem [16]. Although these theories have advanced the state of the field, they could be extended further in order to account for the occurrence of fractional quantum Hall effects at filling factors $\nu > 1$ [25]. In addition, the physical origin of certain peculiar FQHE sequences could be better elucidated, and there is a need to clarify whether these novel quasiparticles at $\nu = 1/2$ maintain their character to, for example, $\nu = 1/3$.

We have also been motivated by evidence of self similarity in the Hall curve and the possibility that this might be more pervasive than what has been revealed thus far by experiment. Thus, this study represents an attempt to extract a repeating pattern, and construct the Hall curve from an 'elementary unit,' in order to identify possible fractal characteristics in quantum Hall effects [1]. Success in this direction might make possible the identification of novel (hidden) FQHE sequences within the main sequence FQHE's in the presently available samples, and perhaps, improve the intuitive understanding of quantum Hall effects.

The Hall curve that is constructed here shows a connection between the occurence of FQHE's at filling factors

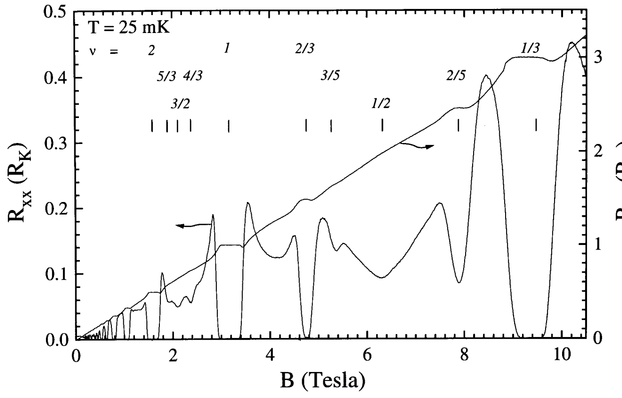


**Fig. 1.** The diagonal resistance, $R_{xx}$, and the Hall resistance, $R_{xy}$, measured in a GaAs/AlGaAs heterostructure device are plotted vs. the magnetic field, B

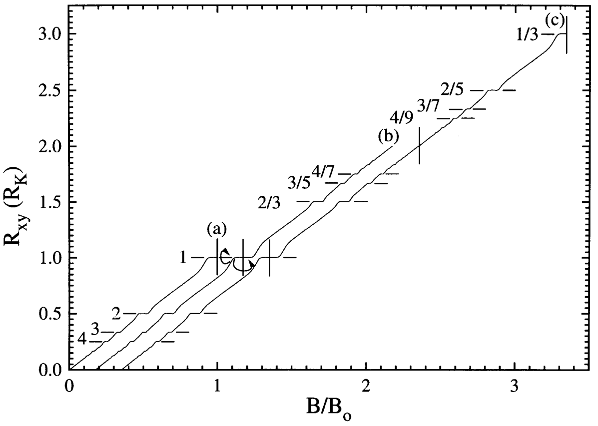


**Fig. 2.** *a* The Hall resistance $R_{xy}$ vs. the normalized magnetic field $B/B_0$ upto filling factor $\nu = 1$ (marked by bold vertical line). *b* To the Hall curve of *a*, one may attach a suitable copy of curve *a*, in order to obtain curve *b*. The copy is obtained by first reflecting the generator about $R_{xy} = 1R_K$, followed by another reflection about $B/B_0 = 1$. *c* to the curve *b*, one may attach a copy of curve *a* after translating it along the B- and $R_{xy}$-axes by two units, respectively. The resultant curve exhibits IQHE's and main sequence FQHE's. Curve *c*, which spans $0 \leq R_{xy} \leq 3R_K$, will be utilized as the template for the following figures. Here, curves *a*, *b*, and *c* have been offset along the abscissa for the sake of clarity

$\nu > 1$ and $\nu < 1$ by demonstrating, for example, that a $\nu = (3p \pm 1)/(4p \pm 1)$ sequence about $\nu = 3/4$, and a $\nu = (3p \pm 1)/(8p \pm 3)$ sequence about $\nu = 3/8$, are reflections of quantum Hall effects about $\nu = 3/2$ in the first Landau band. The construction suggests the existence of a large number of half-filled-Landau-band like neighborhoods (some of which lie in the range $1/2 > \nu > 1/3$) at, for example, $\nu = (4j + 1)/4$, $(4j + 1)/(8j - 2)$, $(4j + 1)/(8j + 6)$, etc., with $j = 1, 2, 3 \ldots$. It also indicates that the QHE structure in the range ($5/6 \geq \nu \geq 3/4$ or $3/4 \geq \nu \geq 1/2$) or ($1/2 \geq \nu \geq 3/8$ or $3/8 \geq \nu \geq 5/14$) is similar to the series which spans a magnetic field range $\Delta B = 2B_0$ with $0 \leq B \leq 2B_0$ or $2B_0 \leq B \leq 4B_0$, where $B(\nu = 1) = B_0$. Finally, apparant similar fractal aspects between FQHE's and the classical problem of an-electron-in-a-periodic-potential-subjected-to-a-magnetic-field indicates the possibility of applying a (single particle) density-of-states type description to the FQHE regime, with fine structure (gaps) within the Landau bands, as in the Hofstadter butterfly spectrum [27–33]. This suggests the understanding that electron transport in a Wigner (poly) crystal tends to result in FQHE's.

The investigation was motivated by experiments carried out on Hall bar type specimens based on GaAs/AlGaAs heterostructures prepared by molecular beam epitaxy [2, 3]. The measurements exhibited FQHE features that have been widely reported in the literature, [13, 24] as is evident in the longitudinal, $R_{xx}$, and Hall resistance, $R_{xy}$, data shown in Fig. 1. The plot shows that between $1/2 \geq \nu \geq 1/3$ the most pronounced resistance minima coincides with a plateau at $R_{xy} = 3R_K$, which characterizes a '1/3' fractional quantum Hall effect; [5] there is also a second deep $R_{xx}$ minimum which corresponds to a '2/5' FHQE. Previous studies have suggested the so-called main sequence series, occuring in the vicinity of $\nu = p/(2p + 1)$ with $p = 1, 2, \ldots$, which correspond to $R_{xy} = 2R_K + R_K/p$ or, equivalently, integral quantum Hall effects shifted by $2R_K$. The main sequence FQHE's in the range $1 \geq \nu \geq 1/2$, of which the '1', '2/3' and the '3/5' are observable in these data, occur at $\nu = p/(2p - 1)$ and correspond to $R_{xy} = 2R_K - R_K/p$. The similarities between the neighborhoods $\nu = 1/2$ and $\nu = \infty$ ($B = 0$) have been formalized by suggesting the existence of a Fermi sea of novel quasiparticles near $\nu = 1/2$, [14–16] such that a deviation about $\nu = 1/2$ is equivalent to deviation about $B = 0$. Thus, a feature of this picture is that a magnetic field $B = 2\phi_0 n$ may be subtracted away in the vicinity of $\nu = 1/2$, where $\phi_0$ is the flux quantum, resulting in an effect that is linearly proportional, with the same coefficient of proportionality in the '+' and '−' B directions, to the remaining magnetic field [14–16].

The basic Hall curve to $\nu = 1/3$ might be constructed (see Fig. 2) from an experimental trace ('fractal generator') showing IQHE upto $\nu = 1$. The construction proceeds by attaching a suitable copy of the elementary unit (curve of Fig. 2a) to the Hall curve of Fig. 2a, in order to obtain the curve shown in Fig. 2b. The copy might be obtained by first reflecting the generator about $B/B_0 = 1$, followed by a reflection about $R_{xy} = 1R_K$ (see Fig. 2). The electron-like main sequence to $\nu = 1/3$ might be realized by attaching the curve of Fig. 2a to the end of the Hall curve of Fig. 2b, after moving the curve of Fig. 2a along the abscissa and ordinate by two units, respectively. Thus, one obtains Fig. 2c which exhibits the main sequence FQHE's to $\nu = 1/3$. In a similar spirit, the Hall curve may be continued to higher Hall resistances and $B/B_0$ by repeatedly concatenating sections. Here, however, the curve of Fig. 2c will be utilized as a template for further constructions shown in the following figures, in order to maintain sufficient resolution in the plots.

A review of the experimental situation shows that FQHE's occur also at $\nu > 1$, for example, about $\nu = 3/2$ (see Fig. 1) [25]. Indeed, the simple relation $\nu = j + \frac{p}{(2p \pm 1)}$ observed between FQHE's in the first Landau band ($j = 1$) and that in the lowest band ($j = 0$) suggests the possibility that FQHE's in higher bands can

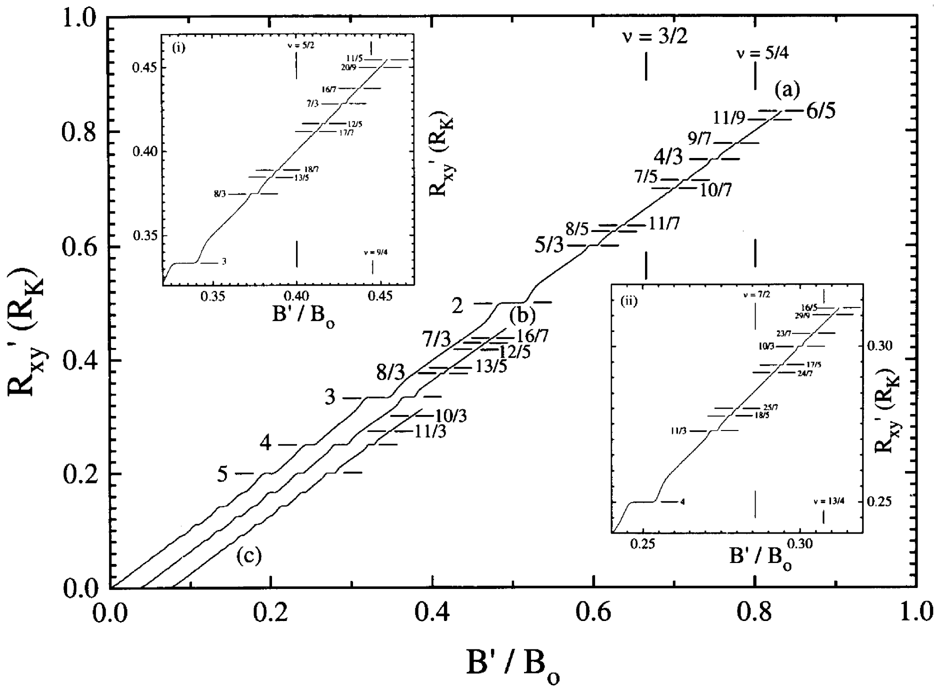

**Fig. 3.** *a* The axes of the template (Fig. 2c) have been rescaled by $B'/B_0 = [2/3 \pm 1/(3(3\ B_0/B \pm 1))]$ and $R_{xy}'/R_K = [2/3 \pm 1/(3(3R_K/R_{xy} \pm 1))]$, respectively. Here, the '+' branch spans the range $3/2 \geq \nu \geq 6/5$. *b* The axes of Fig. 2c have been rescaled by the transformations $B'/B_0 = [2/5 \pm 1/(5(5\ B_0/B \pm 2))]$ and $R_{xy}'/R_K = [2/5 \pm 1/(5(5\ R_K/R_{xy} \pm 2))]$, respectively. This produces additional FQHE's in the range $3 \geq \nu > 2$ which are also shown in inset (i). *c* The axes of Fig. 2c, have been rescaled by the transformations $B'/B_0 = [2/7 \pm 1/(7(7\ B_0/B \pm 3))]$ and $R_{xy}'/R_K = [2/7 \pm 1/(7(7R_K/R_{xy} \pm 3))]$, respectively. The expanded scale of inset (ii) highlights the range $4 \geq \nu > 3$. Curves a, b, and c have been offset along the abscissa

be constructed from the result for the lowest band (to $\nu = 1/3$) by applying a suitable set of transformations to the template (Fig. 2c). We demonstrate this point for $2 \geq \nu > 1$ by simultaneously rescaling the axes of the template (Fig. 2c) by the (continuous) transformation $x' = [2/3 \pm 1/(3(3\ x^{-1} \pm 1))]$, where $x = B/B_0$ and $R_{xy}/R_K$, respectively, with the results shown in Fig. 3a. Here, the electron-like '+' branch spans the range $3/2 \geq \nu \geq 6/5$ and the '1/3' FQHE of Fig. 2c is mapped onto the '6/5' FQHE in curve Fig. 3a. In the '−' branch, the '1' IQHE of the template is transformed into the '2' IQHE in this curve, and the $\nu = 1/2$ neighborhood of Fig. 2c is transformed into $\nu = \infty$ ($B'/B_0 = 0$) in Fig. 3a. A noteworthy point here is that these non-linear transformations produce an electron-like '+' branch which spans a smaller range of magnetic fields (and Hall resistances) than the hole-like '−' branch, unlike the case of the lowest band, discussed previously.

In a similar spirit, the axes of Fig. 2c may be rescaled simultaneously by the transformation $x' = [2/5 \pm 1/(5(5\ x^{-1} \pm 2))]$, in order to obtain scaled copies exhibiting FQHE's in the second Landau band. The results are shown in Fig. 3b, and Fig. 3, inset (i). If one rescales the axes of Fig. 2c, by the transformation $x' = [2/7 \pm 1/(7(7\ x^{-1} \pm 3))]$ where $x = B/B_0$ and $R_{xy}/R_K$, respectively, one obtains FQHE's in the third Landau band (Fig. 3c and inset (ii)). The fractal aspect here is that if one increases the magnification and examines the Hall curve between two consecutive IQHE's one sees once again an elementary (integral quantum) Hall curve.

The next step in the construction involves enforcing the mirror (anti) symmetry in the Hall curve that was shown in Fig. 2b, upon the constructed Hall curves of Fig. 3. Suitable compound transformations which include both the 'map to higher Landau band' (as in Fig. 3) and the 'mirror (anti) symmetry' about $\nu = 1$ features (Fig. 2b) are demonstrated in Fig. 4. In Fig. 4a, the axes of the template (Fig. 2c) have been rescaled simultaneously by the transformation $x' = [4/3 \mp 1/(3(3\ x^{-1} \pm 1))]$, where $x = B/B_0$ and $R_{xy}/R_K$, respectively. Then, the '+' branch is the one that spans the range $6/7 \geq \nu \geq 3/4$. In the '−' branch, the '1' IQHE of the template is transformed into the '2/3' FQHE in this curve, and the $\nu = 1/2$ neighborhood (Fig. 2c) is transformed into $\nu = 1/2$ (Fig. 4a). Analogously, the abscissa and ordinate of Fig. 2c may be rescaled simultaneously by the transformations $x' = [8/5 \mp 1/(5(5\ x^{-1} \pm 2))]$, where $x = B/B_0$ and $R_{xy}/R_K$, respectively, in order to generate new FQHE's in the region $2/3 > \nu \geq 3/5$ (see Fig. 4b and inset (i)). Finally, the axes of Fig. 2c, may be rescaled simultaneously by the transformation $x' = [12/7 \mp 1/(7(7\ x^{-1} \pm 3))]$, in order to produce additional FQHE's in the region $3/5 > \nu \geq 4/7$. These are also shown in expanded scale in Fig. 4, inset (ii). Interestingly, these transformations suggest the possibility of additional FQHE sequences between the main sequence 'hole-like' FQHE's, for example, between the '2/3' and the '3/5' FQHE's (see Fig. 4b), in addition to new $\nu = 1/2$ like neighborhoods between $1 > \nu > 1/2$, e.g., at $\nu = (4j + 1)/(8j - 2) = 5/6,\ 9/14,\ 13/22$, etc., for $j = 1, 2, 3$, etc.

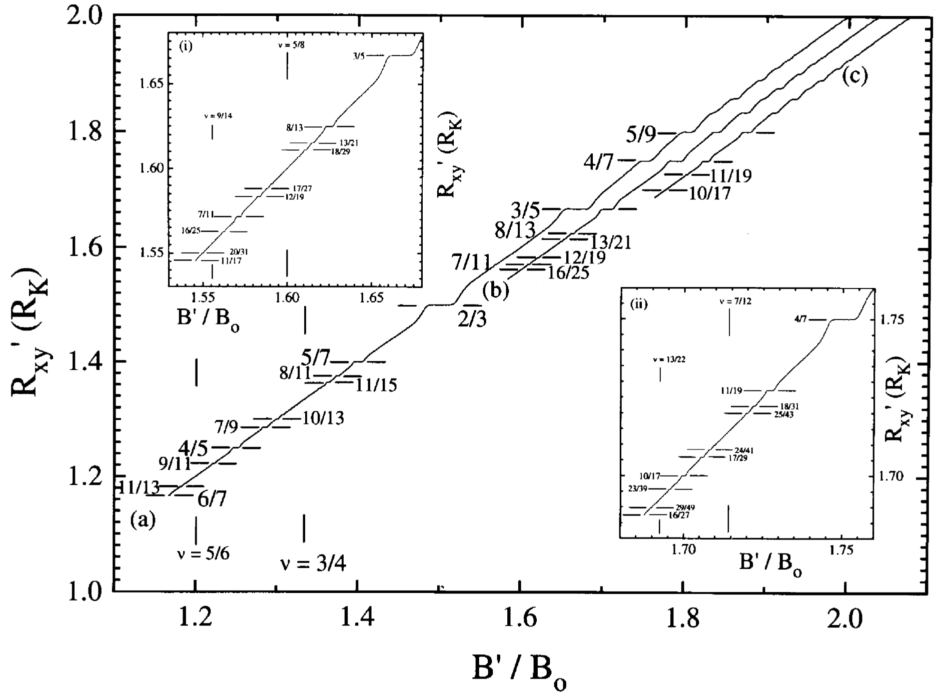


**Fig. 4.** *a* The axes of the template (Fig. 2c) have been rescaled by the transformations $B'/B_0 = [4/3 \mp 1/(3(3\ B_0/B \pm 1))]$ and $R_{xy}'/R_K = [4/3 \mp 1/(3(3\ R_K/R_{xy} \pm 1))]$, respectively. Here, the ' + ' branch spans the range $6/7 \geq \nu \geq 3/4$, and the $\nu = 1/2$ neighborhood (of Fig. 2c) is transformed into the $\nu = 5/6$ neighborhood. In the ' − ' branch, the '1' IQHE of the template (Fig. 2c) is transformed onto the '2/3' FQHE in this curve, and the $\nu = 1/2$ neighborhood is mapped into the $\nu = 1/2$ neighborhood. *b* The abscissa and ordinate of Fig. 2c have been rescaled by the transformations $B'/B_0 = [8/5 \mp 1/(5(5\ B_0/B \pm 2))]$ and $R_{xy}'/R_K = [8/5 \mp 1/(5(5\ R_K/R_{xy} \pm 2))]$, respectively. This produces additional FQHE's in the region $2/3 > \nu \geq 3/5$, see also inset (i). *c* The axes of Fig. 1c, have been rescaled by $B'/B_0 = [12/7 \mp 1/(7(7\ B_0/B \pm 3))]$ and $R_{xy}'/R_K = [12/7 \mp 1/(7(7\ R_K/R_{xy} \pm 3))]$, respectively. For the sake of clarity, curves *a*, *b*, and *c* have been offset along the abscissa

The consequence of FQHE's in higher Landau bands (Fig. 3) for the Hall structure between $1/2 \geq \nu > 1/3$ may be deduced by translating the Hall curves of Fig. 3 by two units along the abscissa and the ordinate, respectively, in analogy to the procedure used for constructing the Hall curve of Fig. 2c from Fig. 2a and Fig. 2b. This is equivalent to, for example, rescaling the axes of the template (Fig. 2c) by the transformation $x' = [8/3 \pm 1/(3(3\,x^{-1} \pm 1))]$, where $x = B/B_0$ and $R_{xy}/R_K$, respectively (see Fig. 5a). Then, the ' + ' branch spans the range $3/8 \geq \nu \geq 6/17$, and the $\nu = 1/2$ neighborhood of Fig. 2c is transformed into the $\nu = 5/14$ neighborhood in the ' + ' branch of Fig. 5a. About $\nu = 3/8$, we expect a 'copy' of IQHE's occuring in order of decreasing strength in the vicinity of filling factors $\nu = (3p \pm 1)/(8p \pm 3)$. In Fig. 5 (b and inset (i)), the axes of Fig. 2c have been rescaled by the transformations $x' = [12/5 \pm 1/(5(5\,x^{-1} \pm 2))]$, where $x = B/B_0$ and $R_{xy}/R_K$, respectively. This produces additional FQHE's in the region $3/7 \geq \nu > 2/5$. Finally, the axes of Fig. 2c, have also been rescaled by the transformations $x' = [16/7 \pm 1/(7(7\,x^{-1} \pm 3))]$, see Fig. 5c. This produces additional FQHE's in the region $4/9 \geq \nu > 3/7$, which are shown in expanded scale in inset (ii).

The complete Hall curve spanning the range $\infty \geq \nu \geq 1/3$ might be obtained by putting together the relevant sections from Figs. 2, 3, 4, and 5. The result is shown in Fig. 6b. This next (second) generation template (Fig. 6b), which ought to be compared with Fig. 6a, exhibits additional FQHE fine structure, including novel FQHE sequences in the lowest band and FQHE in higher Landau bands.

At this stage, it is clear that an iteration has been completed and that this second generation template (Fig. 6b) might now be subjected to a new round of transformations as above in order to obtain a third generation template. Then, the third generation template may be similarly operated upon in order to obtain a fourth generation template, and so on [1].

A continued (iterative) application of this prescription seems to produce ever finer Hall structure and, simultaneously, provides some new insight into origin of the relative strength- and heirarchy- of FQHE sequences: First, it is clear that the form of the transformation themselves influence the relative strength of the plateaus. This may be understood by comparing Fig. 3a and Fig. 3b and noting that FQHE plateaus lying between $3 \geq \nu > 2$ are narrower than those lying between $2 \geq \nu > 1$ although they both result from the application of a single transformation upon the same template (Fig. 2c). Second, compounding transformations further narrows a given Hall plateau while giving rise to new QHE's. (This is somewhat analogous to the typical experimental observation that integral Hall plateaus 'become narrower' as fractional Hall plateaus become observable with improving sample quality). Further, the larger the number of iterative loops that is necessary in order to realize a particular sequence, the weaker the resultant sequence. Finally, the figures

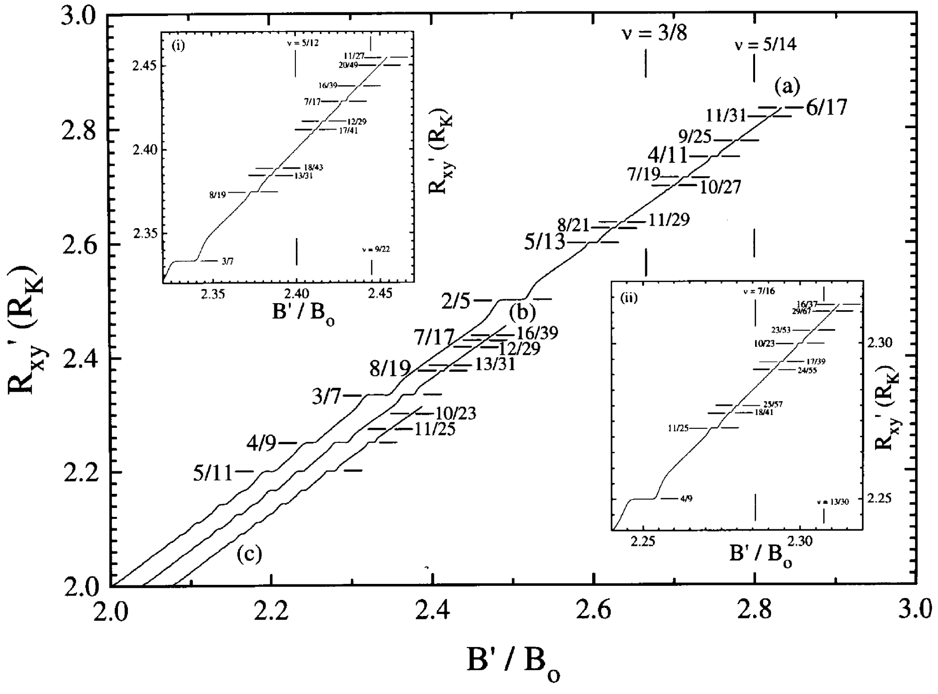


**Fig. 5.** *a* The axes of the template (Fig. 2c) have been rescaled by the transformations $B'/B_0 = [8/3 \pm 1/(3(3\ B_0/B \pm 1))]$ and $R_{xy}'/R_K = [8/3 \pm 1/(3(3R_K/R_{xy} \pm 1))]$, respectively. Here, the '+' branch spans the range $3/8 \geq \nu \geq 6/17$, and the '1/3' FQHE from Fig. 2c is mapped onto the '6/17' FQHE in curve **a**. In addition, the $\nu = 1/2$ neighborhood is mapped onto the $\nu = 5/14$ neighborhood. In the '−' branch, the '1' IQHE of the template is transformed into the '2/5' FQHE in this curve. **b** The axes of Fig. 2c have been rescaled by the transformations $B'/B_0 = [12/5 \pm 1/(5(5\ B_0/B \pm 2))]$ and $R_{xy}'/R_K = [12/5 \pm 1/(5(5\ R_K/R_{xy} \pm 2))]$, respectively. This produces additional FQHE's in the region $3/7 \geq \nu > 2/5$, which is shown in expanded scale in inset (i). *c* The axes of Fig. 2c, have been rescaled by the transformations $B'/B_0 = [16/7 \pm 1/(7(7\ B_0/B \pm 3))]$ and $R_{xy}'/R_K = [16/7 \pm 1/(7(7\ R_K/R_{xy} \pm 3))]$, respectively. This produces additional FQHE's in the region $4/9 \geq \nu > 3/7$, which is shown in expanded scale in inset (ii). Note that curves *a*, *b*, and *c* have been offset along the abscissa

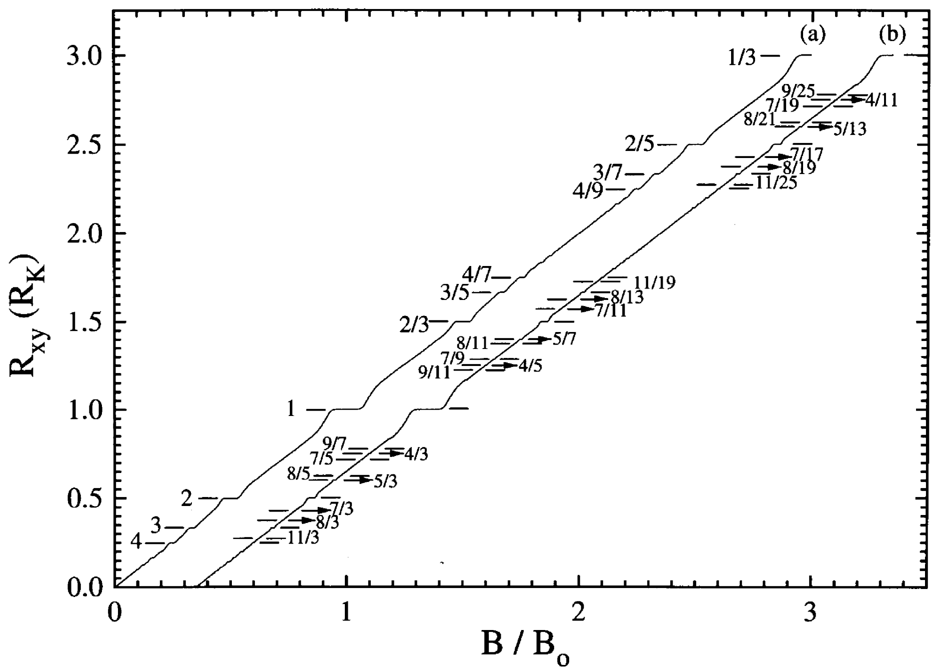


**Fig. 6** *a* A copy of the 'template' Hall curve shown in Fig. 2c. *b* The next (second) generation 'template' Hall curve obtained by splicing together the relevant regions from the Hall curves shown in Figs. 2, 3, 4, and 5. This curve *b*, which has been offset along the abscissa, exhibits FQHE in higher Landau bands and novel FQHE's in the lowest band. One might imagine subjecting this second generation curve to a new round of transformations (see text) in order to obtain a third generation template

show that some QHE's appear repeatedly in several different transformed curves. This indicates that the greater the number of transformations that help realize a given QHE, the stronger the resultant QHE. There is some evidence that these QHE's, which may be obtained in many different ways, are also most readily observed in experiment.

The existence of a prescription for constructing the full Hall curve from an elementary unit, and the fact that fine structure in the Hall curve may be related to a suitably scaled version of the elementary unit, clearly demonstrate the existence of fractal characteristics in QHE's [1]. Further, if one cites the occurence of IQHE's as a manifestation of mobility gaps in the single particle spectrum, then one might reason that fine structure in the Hall curve signifies also the existence of gaps due to a fractal electronic structure within the Landau band, analogous to that in the Hofstadter butterfly spectrum [27–34]. The literature connected with this periodic potential problem [27–34] typically examines the electronic structure of a two-dimensional square lattice immersed in a uniform transverse magnetic field, and the influential parameter in the analysis is $\alpha$, the ratio of the flux through a lattice cell to $\phi_0$, which is taken to satisfy the condition of rationality, $\alpha = m/q$. When $\alpha = m/q$, the Bloch band breaks up into q distinct energy bands. Equivalently, the Landau band is broken into m bands by the periodic potential [31]. One might link this model (periodic) system with the system under consideration, a homogeneous 2DES, by suggesting that the average interelectronic spacing itself sets an effective length scale for a periodic potential in the 2DES (because each electron sees a small periodic variation in the background potential due to the existence of other electrons), and this periodicity is responsible for a fractal electronic (fine) structure within the Landau band. Thus, for example, at $\alpha = 3/1$, when there occurs three flux quanta per unit cell (of the electron lattice), the Landau band might be split into three subbands by the periodic potential (see Fig. 7a). As three flux quanta per electron corresponds to a one-third filled Landau band, the Fermi level would be pinned within the lowest mobility gap (see Fig. 7a) and Hall quantization appears plausible at $R_{xy} = 3R_K$. If $\alpha = 3/2$, the lowest Landau band might be split, once again, into three subbands. However, in this case, the Fermi level would lie in the gap between the second and the third subbands because $\nu = 2/3$. Then, Hall quantization appears plausible at $R_{xy} = 3/2\, R_K$.

In Fig. 7b, we examine some 'numerology' for odd numerator rational fractions $\alpha = m/q$ with m upto $m = 13$. That is, we wish to determine whether all such FQHE's at $R_{xy} = (m/q)\ R_K$ predicted by this simple picture occur in the Hall curve construction discussed here. In order to keep track of FQHE's that occur, observed fractional values of $\alpha$ have been linked into sequences in Fig. 7b. Indeed, a close study of the constructed Hall curves shows that all FQHE's corresponding to $R_{xy} = (m/q)\ R_K$ with $1 \le m \le 11$ and $1 \le q \le 11$ do occur in such a construction, which suggests the conjecture that the entire set of rational numbers $\alpha = m/q$, with m assuming odd-integral values and q assuming integral values can, in principle, be manifested in a such a constructed Hall curve exhibiting FQHE's (at $\nu = q/m$).

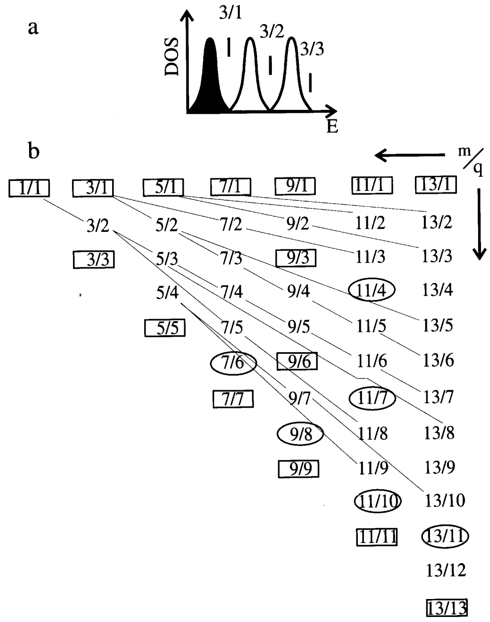


**Fig. 7a.** A sketch exhibiting the splitting of a Landau band into three subbands when the flux per electron $\alpha = (m/q) = (3/q)$ with $q = 1$, 2, and 3. Here, $q = 1$ corresponds to three flux per electron or $\nu = 1/3$, which may be identified with the filling of the lowest subband (shaded region). Similarly, $q = 2$ corresponds to two filled subbands or $\nu = 2/3$, and $q = 3$ corresponds to three filled subbands or $\nu = 1$. **b** The set of rational fractions m/q, with odd-integral m, $1 \le m \le 13$, and $1 \le q \le 13$. Fractional quantum Hall effects in the lowest band, with Hall resistances $R_{xy} = (m/q)\ R_K$, that occur in this construction have been joined together into sequences. Simple fractions have been highlighted by rectangular frames. Fractions that do not fall into a simple sequence (with this restricted set of rational fractional numbers), and yet occur in such a construction, are framed by ovals. One observes that all odd numerator rational fractions m/q with $1 \le m \le 11$ and $1 \le q \le 11$ are represented in such a construction. This suggests the conjecture that every odd-numerator rational fraction (m/q) could, in principle, be represented by a FQHE in a Hall curve constructed using the method outlined in the text

The Hofstader type picture of a Landau band being split into an odd-integral number of subbands by a commensurate magnetic field suggests a dynamical structure for the gaps in the density of states as a function of $\nu$. Yet, the point that one always obtains a gap at the Fermi level at odd denominator (rational) fractional $\nu$ ($\nu = q/m$) suggests the possibility of utilizing a static picture, one that includes a number of mobility gaps occuring at various energies within the Landau band, a picture that is more easily grasped by the intuition. Such a static picture including mobility gaps within the Landau band is shown in Fig. 8. Figure 8a shows the usual picture for IQHE including mobility gaps at $E = p\hbar\omega$. This picture is extended to the $\nu < 1$ fractional filling factor QHE regime in Fig. 8b. Here, one sees mobility gaps occuring within the lowest Landau band at $E = \hbar\omega/[2 \pm (1/p)]$, which correspond to the main sequence FQHE's. Similarly, fractal electronic structure in the range $2 > \nu > 1$ might also be represented by gaps over the corresponding range, as in Fig. 8c.

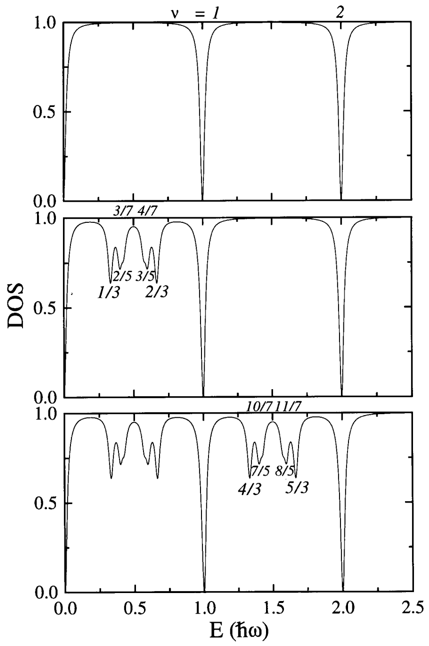


**Fig. 8.** A sketch of the 'extended' density of states vs. energy. *a* The usual situation which gives IQHE when the Fermi level lies the mobility gaps in the vicinity of $E = p\hbar\omega$. *b* Suggested fractal structure in the electronic spectrum (see text) indicates additional mobility gaps within the Landau band. Here, in this 'static picture,' gaps are indicated at filling factors corresponding to main sequence FQHE's in the lowest Landau band. *c* A similar picture for FQHE's in the filling factor range $1 < \nu < 2$

Thus, the overview that emerges from this study is as follows: FQHE's are a transport response which become observable when an electron becomes sensitive to the periodic arrangement of other electrons. The reasoning here is that a lattice-like electron arrangement induces a fractal electronic structure with gaps within the Landau band, and it is the pinning of the Fermi level within these gaps that induces FQHE's. A consequence of such reasoning is that a correlated electron arrangement occurs not only under FQHE conditions but also in the range of $\nu$ between consecutive FQHE's (it is just that the transport response does not reflect this periodicity in a very obvious way under such conditions). Indeed, observability of FQHE at the highest $\nu$, seems tantamount to confirming this (local) Wigner crystallization for lower $\nu$.

In summary, an iterative prescription has been specified here for constructing the Hall curve – one that generates FQHE's in higher Landau bands and also identifies new sequences between 'main sequence' FQHE's. The analogies between this Hall curve construction and the construction of fractals [1] was pointed out throughout the procedure, and the self-similarity between the various sections of the overall Hall curve and the template (Fig. 2c) was identified with nonlinear transformations to be applied simultaneously to the $R_{xy}$ and B axis. The development of the fractal Hall curve suggested the existence of a number of half-filled-Landau-band-like-neighborhoods (at, for example, $\nu = (4j + 1)/(8j + 6)$) which interfere between the $\nu = 1/2$ neighborhood and $\nu = 1/3$ FQHE's; this seems to complicate the question whether novel quasiparticles at $\nu = 1/2$ can maintain their character to $\nu = 1/3$. Finally, it has been reasoned that FQHE's constitute a fractal originating from a Hofstader type spectrum, that is induced by (local) Wigner crystallization.

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