Reflectance on conducting media | Chapt. 8 Waveguides...

If we look at boundaries between an ideal conductor and an Isolator, we have inside of a conductor $\vec{E}=0$.

$\rightsquigarrow$ EM-wave in insulator has to vanish on the interface.

$\rightarrow$ A time-varying EM-field will produce a surface-charge density to give zero electric field inside.

↳ <u>Boundary conditions:</u>

① $\oint \vec{D}\cdot d\vec{f} = Q = \oint \sigma\, df$

↳ $\vec{D}\cdot\hat{n} = \sigma$

σ: surface charge density on conductor surface

$\hat{n}$: normal vector

② $\oint \vec{H}\cdot d\vec{s} \approx H_{I_{\text{solator}}}\cdot\hat{t} - H_{C_{(\text{conductor})}}\cdot\hat{t} = \int(\vec{\nabla}\times\vec{H})\cdot d\vec{f}$

$$= \int(\underbrace{\dot{\vec{D}}} + \vec{j})\cdot d\vec{f} = \int \vec{j}\cdot d\vec{f} = J$$

└ on surface $\vec{E}=0 \rightsquigarrow \dot{\vec{D}}=0$

From $\vec{H}_C = 0$, we get $\vec{H}_I\cdot\hat{t} = J$: surface current

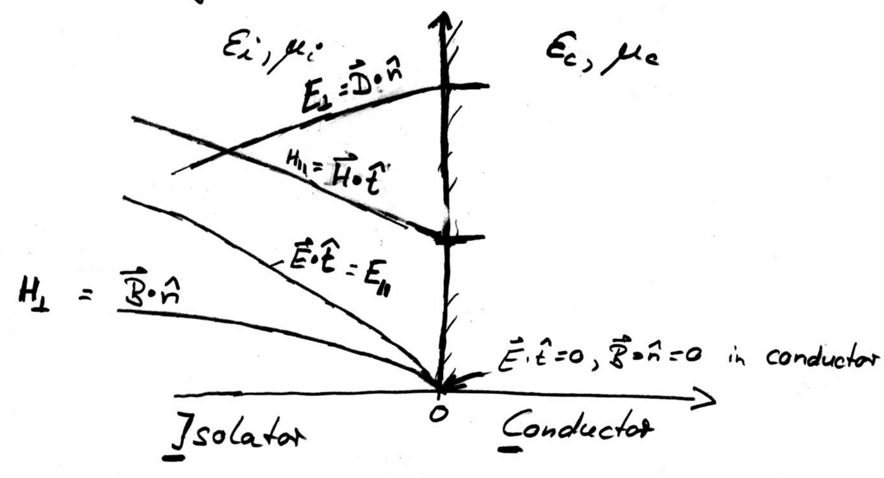
$\vec{D}\cdot\hat{n}$ and $\vec{H}\cdot\hat{t}$ are <u>not continuous</u> on Conductor surface

but $\vec{E}\cdot\hat{t}$ and $\vec{B}\cdot\hat{n}$ <u>are continuous</u>

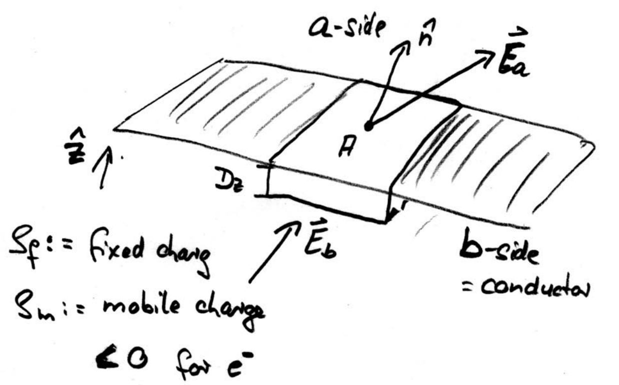
Since inside of the conductor the field have to vanish, we have

$$\boxed{\vec{E}\cdot\hat{t}=0} \quad \text{and} \quad \boxed{\vec{B}\cdot\hat{n}=0}$$

On the Interface Conductor - Isolator we have the following behavior:



Remember: in EM-I, we learned already that for a real conductor or semiconductor the condition of a surface charge of zero thickness had to be modified:



$$\int_S d^2r\ \hat{n}\cdot\vec{E} = 4\pi\int_V d^3r\,[\rho_f+\rho_m]$$

$$= A\cdot\hat{z}\cdot\vec{E}_a\Big|_{a\text{-side}} + A\cdot(-\hat{z})\cdot E_b\Big|_{\substack{\text{depth } D_z\\ \text{within}\\ \text{conductor}}}$$

$$= 4\pi A\int_{-\Delta z}^{0} dz\,[\rho_f(z)+\rho_m(z)]$$

note: $\rho_f + \rho_m = 0$ in bulk of conductor!

side contributions cancel in pairs (the normal vectors of opposite sides point in opposite directions)

take Δz deep enough so that $\vec{E}_b$ is effectively zero

$$\rightsquigarrow\ \hat{z}\,\vec{E}_a = 4\pi\sigma = 4\pi\int_{-\Delta z}^{0^+} dz\,(\rho_f(z)+\rho_m(z))$$

$\rightsquigarrow$ definition of surface charge density σ! $\Longrightarrow$

$\rightsquigarrow$ from this we got the "skin depth" δ.

$\hookrightarrow$ for metals: $L_{TF} = \sqrt{\frac{\pi E_F}{2e^2 n_0}}$, Thomas-Fermi screening length

E_F : Fermi energy

$n_0 \sim 10^{23} cm^{-3}$ mobile charge

Since the $\vec{E}$-field penetrates the surface $\rightsquigarrow$ the surface current @ interface has to be modified: using ohms law

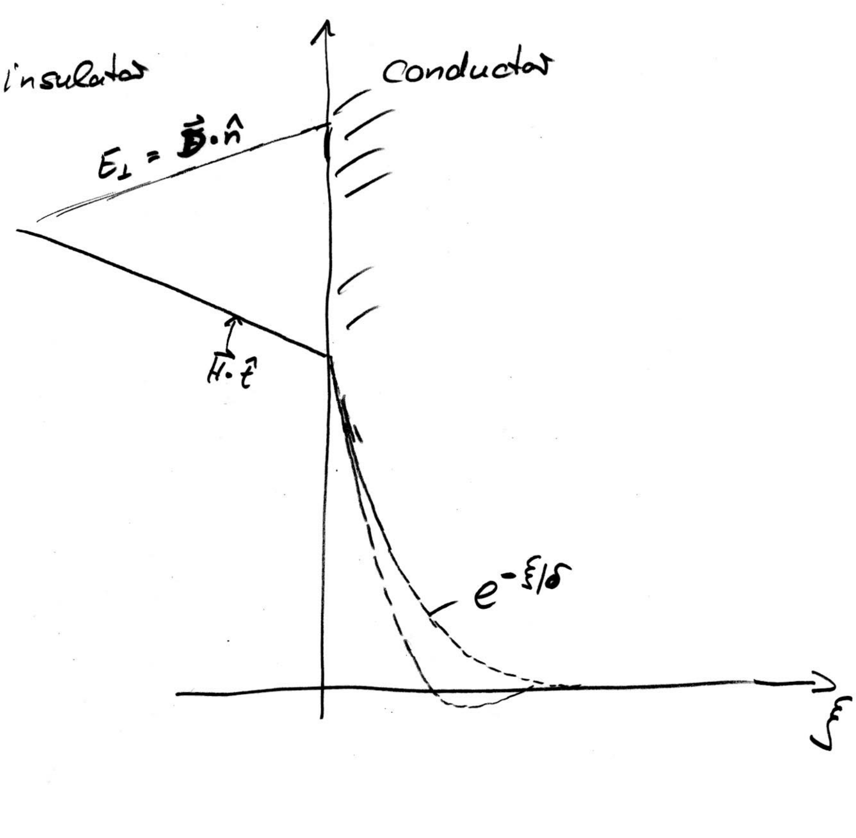
$$j = \sigma E \qquad \text{with } \sigma = \int_{-\Delta z}^{0^+} (\rho_{f} + \rho_m(z))\, dz$$

as the current flow through the skin deepth region!

see Jackson 8.1 (p. 353-56) ; $\rightsquigarrow n \times (H - H_c) = 0$, H_c: magnetic field in conductor

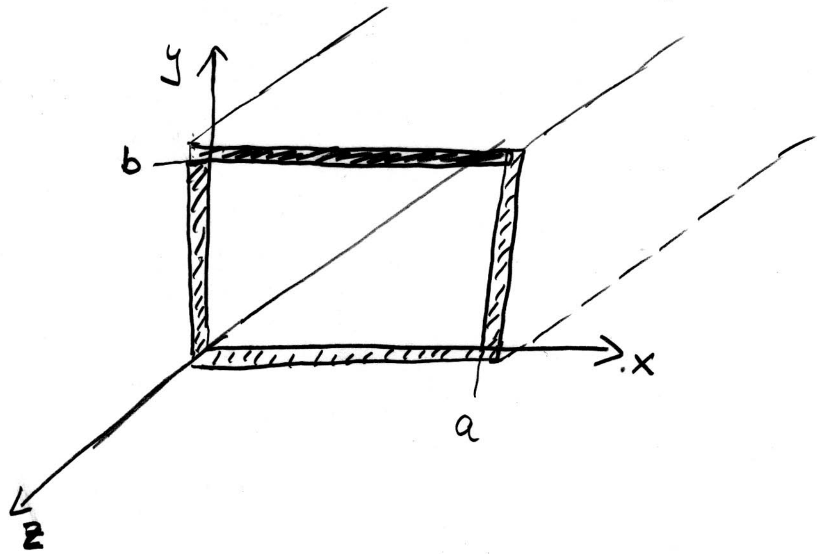
$$\hookrightarrow j = \sigma E = \frac{1}{\delta}(1-i)(\hat{n} \times \vec{H}_{\parallel})\, e^{-\xi(1-i)/\delta}$$, δ: skin depth

ξ: normal coord. inwards to conductor



Example: <u>Rectangular Waveguide</u>

(see cylindrical waveguide in Jackson 8.2, p. 356 → Homework!)



Assume a rectangular conductor with the inside length a, b in x- and y- direction respectively.
The z-axis lays parallel to the conductor axis/tube.

The Maxwell Equations in the inside of the tube are:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad ; \quad \vec{\nabla} \cdot \vec{E} = 0$$

$$\nabla \times \vec{B} = \mu \varepsilon \frac{\partial \vec{E}}{\partial t} \quad \text{and} \quad \nabla \cdot \vec{B} = 0$$

with the boundary values:

$$x = 0, a \quad : \quad E_y = 0 \;,\; E_z = 0 \;,\; B_x = 0$$

$$y = 0, b \quad : \quad E_x = 0 \;,\; E_z = 0 \;,\; B_y = 0$$

subject to Telegraph equation for a non-conducting media ($\sigma = 0$):

DE: $$\boxed{\Delta \vec{E} - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \vec{E} = 0} \quad \text{or} \quad \boxed{\Delta \vec{B} - \frac{1}{v^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0}$$

with $\boxed{\frac{1}{v^2} = \varepsilon \cdot \mu}$

⇒

↝ Find the solution of the wave equation in form of a planar wave, propagating in z-direction:

$$\vec{E}(\vec{r},t) = \vec{E}(x,y) \cdot e^{i(k \cdot z - \omega t)}$$

$$\vec{B}(\vec{r},t) = \vec{B}(x,y) \cdot e^{i(k \cdot z - \omega t)}$$

put trail solution in DE and find

$$\Delta_z \vec{E} + \left(\frac{\omega^2}{v^2} - k^2\right)\vec{E} = 0 \qquad \Big| \quad \Delta_z = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\Delta_z \vec{B} + \left(\frac{\omega^2}{v^2} - k^2\right)\vec{B} = 0$$

A variety of solutions are possible for the rectangular conducting tube, which can be narrowed down with additional boundary conditions:

1.) $B_z = 0$ anywhere inside the tube

↳ TM-wave (Transverse Magnetic)

or $E_z|_S = 0$ from $\hat{n} \times \vec{E}_z = 0$, ($n$: unit vector on surface)

2.) $E_z = 0$ anywhere inside the tube

↳ transverse-electric (TE) wave

or $\left.\frac{\partial B_z}{\partial \hat{n}}\right|_S = 0$, $\frac{\partial}{\partial n}$: normal derivative at a point on the surface ⟹

The different waves can be experimentally generated, depending on the excitation mechanism.

Let's look in the following to the TE-waves for which we had: $E_z = 0$ anywhere in the tube.

Find first the z-component of the $\vec{B}$-field B_z:

$\rightsquigarrow$ for $x = [0, a]$: $E_y = 0$

$$\hookrightarrow \quad (\vec{\nabla} \times \vec{B})_y = -i\omega\mu\varepsilon E_y = 0$$

$$\hookrightarrow \quad \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = 0$$

Since $\vec{B} = \vec{B}(x,y)$ we find $\boxed{\dfrac{\partial B_z}{\partial z} = 0}$

which means, the boundary value for a TE-wave

is $\dfrac{\partial B_z}{\partial z} = 0$ for $x = [0, a]$

The solution of the DE for B_z with the above Boundary value can be found with a separation trail

$$B_z(x,y) = f(x) \cdot g(y)$$

put it in DE:

$$\frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial y^2} + \left(\frac{\omega^2}{v^2} - k^2\right) B_z = 0 \qquad \Longrightarrow$$

$\rightsquigarrow$ an get:

$$\frac{f''}{f} + \frac{g''}{g} + \left(\frac{\omega^2}{v^2} - k^2\right) = 0 \qquad \Bigg| \quad f'' = \frac{\partial^2 f}{\partial x^2} \quad g'' = \frac{\partial^2 g}{\partial y^2}$$

from which we get the two DE:

$$\boxed{f'' + \alpha^2 f = 0} \quad \text{and} \quad \boxed{g'' + \beta^2 f = 0}$$

with: $\boxed{-\alpha^2 - \beta^2 + \frac{\omega^2}{v^2} - k^2 = 0}$ – Dispersion relation

From the boundary values follows:

$x = [0, a] \rightsquigarrow \frac{\partial f}{\partial x} = 0$

$y = [0, b] \rightsquigarrow \frac{\partial g}{\partial y} = 0$

For the general solution of the DE

$$f(x) = A \cdot \cos(\alpha x) + B \cdot \sin(\alpha \cdot x)$$

$$\hookrightarrow \frac{\partial f}{\partial x} = -\alpha \cdot A \cdot \sin(\alpha \cdot x) + \alpha \cdot B \cdot \cos(\alpha \cdot x)$$

$\left.\frac{\partial f}{\partial x}\right|_{x=0} = 0 \rightsquigarrow B = 0$; $\left.\frac{\partial f}{\partial x}\right|_{x=a} = 0 \rightsquigarrow \sin \alpha \cdot a = 0$

$\hookrightarrow \alpha = \frac{\pi \cdot n}{a}$

$$\hookrightarrow \boxed{f(x) = A \cdot \cos\left(\frac{n \cdot \pi}{\alpha} \cdot x\right)}$$

$\Rightarrow$

Similarly for $g(y)$:

$$g(y) = A \cdot \cos(\beta \cdot y) + B \cdot \sin(\beta \cdot y)$$

$$\frac{\partial}{\partial y} g(y) = -\beta \cdot A \cdot \sin(\beta \cdot y) + \beta \cdot B \cdot \cos(\beta \cdot y)$$

$$\left.\frac{\partial g}{\partial y}\right|_{y=0} = 0 \rightsquigarrow B = 0, \quad \left.\frac{\partial g}{\partial y}\right|_{y=b} \rightsquigarrow \sin(\beta \cdot y) = 0 \rightsquigarrow \beta = \frac{\pi \cdot m}{b}$$

$$\hookrightarrow \boxed{g(y) = A' \cdot \cos\left(\frac{\pi \cdot m}{b} \cdot y\right)}$$

Integrating the values for α, β in dispersion relation:

$$-\alpha^2 - \beta^2 + \frac{\omega^2}{v^2} - k^2 = 0 \Rightarrow k^2 = \frac{\omega^2}{v^2} - \frac{n^2 \cdot \pi^2}{a^2} - \frac{m^2 \pi^2}{b^2}$$

or $\quad \omega_{nm} = v \cdot \pi \cdot \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}} \quad , \quad n, m \in \mathbb{N} \geq 0$

$$\boxed{k = \frac{1}{v} \cdot \sqrt{\omega^2 - \omega_{nm}^2}} \quad \underline{\text{Dispersion relation}}$$

An undamped EM-wave is obtained for k <u>real</u>

which means $\boxed{\omega > \omega_{nm}}$

If we assume for our rectangular tube $a > b$, than the smallest possible frequency for an undamped wave is given by $m = 0$, $n = 1$:

$$\omega_{10} = \frac{v \cdot \pi}{a}, \quad k = \sqrt{\frac{\omega^2}{v^2} - \frac{\pi^2}{a^2}}$$

$\rightsquigarrow$ smallest possible frequency is given by the largest side in rectangular tube!

$\Rightarrow$

The complete solution for the z-components of the field is

$$B_z = B_0 \cos\left(\frac{n\pi}{a}\cdot x\right)\cos\left(\frac{m\pi}{b}\cdot y\right)\cdot \exp[i(k\cdot z-\omega t]$$

$$E_z = 0$$

For the other components of the fields: E_x, E_y, B_x and B_y - we just calculate them for the base frequency ω_{10}:

ω_{10} $(n=1, m=0)$:
$$E_z = 0$$
$$B_z = B_{0z}\cdot\cos\left(\frac{\pi}{a}x\right)\cdot e^{i(k\cdot z-\omega t)}$$

Let's fix the excitation mechanism with $E_x = 0$, than we get from $\vec{\nabla}\times\vec{E} = -i\omega\vec{B} \rightsquigarrow \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z$

or $\frac{\partial E_y}{\partial x} = i\omega B_z$

$$\hookrightarrow E_y = \frac{i\omega a}{\pi} B_{0z}\sin\left(\frac{\pi x}{a}\right)\cdot e^{i(k\cdot z-\omega t)}$$

For the $\vec{B}$-field, we have

$$\vec{\nabla}\times\vec{B} = -\frac{i\omega}{c^2}\cdot\vec{E}$$

from $E_x = 0 \rightsquigarrow \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = 0 \rightsquigarrow \frac{\partial B_y}{\partial z} = 0$ | since $\frac{\partial B_z}{\partial y} = 0$

$\hookrightarrow B_y = \text{constant}$

Since we only look at time-varying fields, we can set $B_y = 0$.

$\Longrightarrow$

Also from the Maxwell-Ampere Law we find

$$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = -i\frac{\omega}{v^2} E_y$$

$$\hookrightarrow \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} - i\frac{\omega}{v^2} E_y$$

$$= \frac{a}{\pi}\left(\frac{\omega^2}{v^2} - \frac{\pi^2}{a^2}\right)\cdot B_{0z} \sin\left(\frac{\pi}{a}\cdot x\right)\cdot e^{i(k\cdot z-\omega t)}$$

$$\hookrightarrow B_x = -i\,\frac{a\cdot k}{\pi} B_{0z} \sin\left(\frac{\pi}{a}\cdot x\right)\cdot e^{i(k\cdot z-\omega t)}$$

With this we have all six field components for $\vec{E}$ and $\vec{B}$.

The experimental observed field are the real-part of the complex field:

$$\boxed{\begin{aligned}
E_x &= 0 \\
E_y &= -\frac{\omega a}{\pi} B_0 \sin\left(\frac{\pi}{a}\cdot x\right) \sin(k\cdot z - \omega t) \\
E_z &= 0 \\
B_x &= \frac{a\cdot k}{\pi} B_0 \sin\left(\frac{\pi}{a} x\right) \sin(k\cdot z - \omega t) \\
B_y &= 0 \\
B_z &= B_0 \cos\left(\frac{\pi}{a} x\right)\cdot \cos(k\cdot z - \omega t)
\end{aligned}}$$

Since all field are independent of y, the fields have along the y-axis all the same Elongation.