EM. - waves and Wave Propagation:

Reflection and Refraction on Plane Interfaces

Each medium is characterized by a unique set of $\varepsilon(\omega), \mu(\omega)$!

↳ entering / leaving a medium → change of $\varepsilon(\omega)$ and/or $\mu(\omega)$

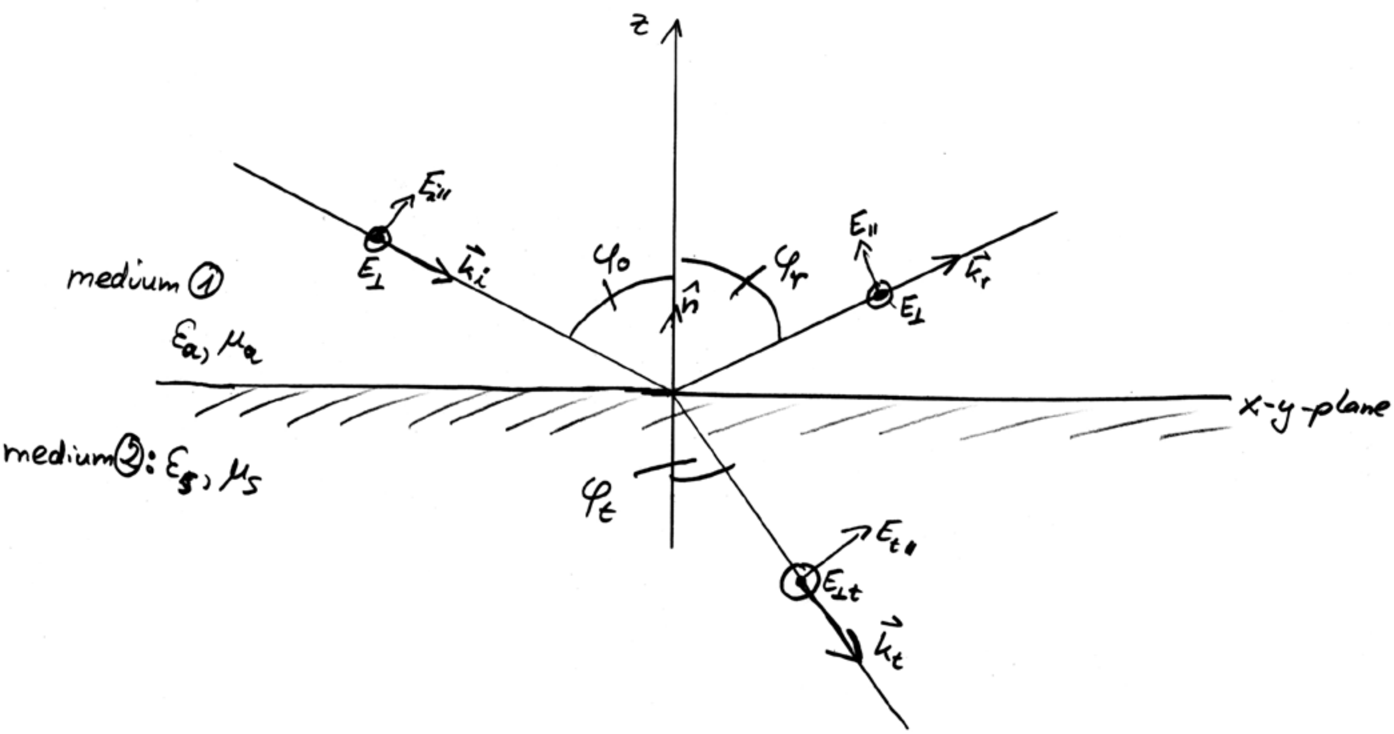
↳ Assume an incidence plane EM-wave:

$$\vec{E}_i(\vec{r},t) = \vec{E}_{oi} \exp\{i(\vec{k}_o\vec{r} - \omega t)\}$$

and

$$\vec{B}_i(\vec{r},t) = \vec{B}_{oi} \exp\{i(\vec{k}_o\vec{r} - \omega t)\}$$

impinging an interface - separating two media, each characterized by a unique set ε, μ:



a.) Plane of incidence is defined by $\vec{k}_i$ and $\hat{n}$

b.) Parallel polarized light: $\vec{E}$ - vector in plane of incidence

c.) Perpendicular polarized light: $\vec{E}$ - vector $\perp$ plane of incidence ⟹

$\rightsquigarrow$ The refracted and reflected EH-waves are given as

$$\vec{E}_t(\vec{r},t) = \vec{E}_{t_0} \exp[i(\vec{k}_t \cdot \vec{r} - \omega t)] \quad , \quad \vec{B}_t(\vec{r},t) = \vec{B}_{t_0} \exp[i(\vec{k}_t \cdot \vec{r} - \omega t)]$$

$$\vec{E}_r(\vec{r},t) = \vec{E}_{r_0} \exp[i(\vec{k}_r \cdot \vec{r} - \omega t)] , \quad \vec{B}_r(\vec{r},t) = \vec{B}_{r_0} \exp[i(\vec{k}_r \cdot \vec{r} - \omega t)]$$

Now, apply Boundary conditions:

a.) spatial variation @ $z=0$ are the same : phase factors $|_{z=0}$ = same

$$\rightsquigarrow \quad \vec{k}_i \cdot \vec{r}\Big|_{z=0} = \vec{k}_r \cdot \vec{r}\Big|_{z=0} = \vec{k}_t \cdot \vec{r}\Big|_{z=0}$$

$\hookrightarrow$ $k_t \cdot \hat{y} = k_r \cdot \hat{y} = 0 \quad \Rightarrow$ reflected & refracted wave are in the plane of incidence!

$\hookrightarrow$ $|\vec{k}_i| \cdot \sin\varphi_0 = |\vec{k}_t| \cdot \sin\varphi_t = |\vec{k}_r| \cdot \sin\varphi_r$ and $|\vec{k}_i| = |\vec{k}_r|$

from which we get $\sin\varphi_0 = \sin\varphi_r$ or $\boxed{\varphi_0 = \varphi_r}$

Law of reflection

from $|\vec{k}_i| \cdot \sin\varphi_0 = |\vec{k}_t| \cdot \sin\varphi_t$

$$\rightsquigarrow \quad \frac{\sin\varphi_0}{\sin\varphi_t} = \frac{|\vec{k}_t|}{|\vec{k}_i|} = \frac{\omega/c \cdot \sqrt{\varepsilon_s \cdot \mu_s}}{\omega/c \sqrt{\varepsilon_a \mu_a}} = \frac{n_s}{n_a}$$

$$\text{or} \quad \boxed{\frac{\sin\varphi_0}{\sin\varphi_t} = \frac{n_s}{n_a}}$$

Law of refraction
(Snell's Law)

b) Maxwell' boundary conditions:

- normal components of $\vec{D}$ and $\vec{B}$ are continuous @ $z=0$
- tangential components of $\vec{E}$ and $\vec{H}$ are " "

$\Longrightarrow$

$$① \quad \left[\vec{D}_{i_0} + \vec{D}_{r_0} - \vec{D}_{t_0}\right] \cdot \hat{n} = 0 = \left[\varepsilon_a\left(\vec{E}_{i_0} + \vec{E}_{r_0}\right) - \varepsilon_s \vec{E}_{t_0}\right] \cdot \hat{n}$$

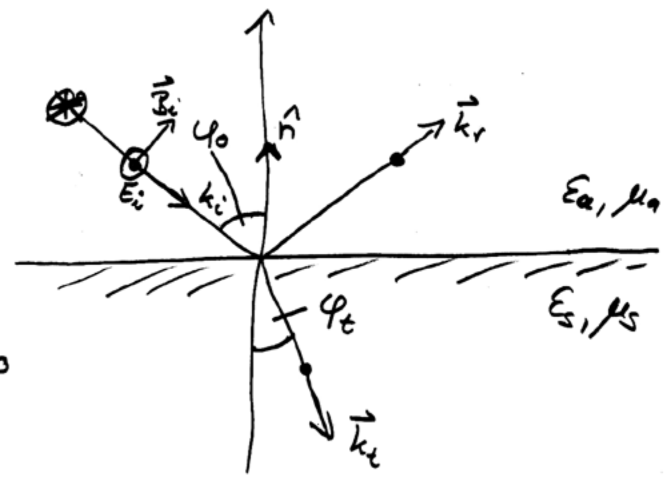
$$② \quad \left[\vec{B}_{i_0} + \vec{B}_{r_0} - \vec{B}_{t_0}\right] \cdot \hat{n} = 0 = \left[\vec{k}_i \times \vec{E}_{i_0} + \vec{k}_r \times \vec{E}_{r_0} - \vec{k}_t \times \vec{E}_{t_0}\right] \cdot \hat{n}$$

and

$$③ \quad \left[\vec{E}_{i_0} + \vec{E}_{r_0} - \vec{E}_{t_0}\right] \times \hat{n} = 0$$

$$④ \quad \left[\vec{H}_{i_0} + \vec{H}_{r_0} - \vec{H}_{t_0}\right] \times \hat{n} = 0 = \left[\frac{1}{\mu_a}\left(\vec{k}_i \times \vec{E}_{i_0}\right) + \frac{1}{\mu_a}\left(\vec{k}_r \times \vec{E}_{r_0}\right) - \frac{1}{\mu_s}\left(\vec{k}_t \times \vec{E}_{t_0}\right)\right] \times \hat{n}$$

Let's first consider EM-wave vector $\perp$ (perpendicular) to the plane of incidence



from ③:

$$\vec{E}_{i_0} + \vec{E}_{r_0} - \vec{E}_{t_0} = 0$$

from ④:

$$\sqrt{\frac{\varepsilon_a}{\mu_a}}\left(|\vec{E}_{i_0}| - |\vec{E}_{r_0}|\right) \cdot \cos\varphi_0 - \sqrt{\frac{\varepsilon_s}{\mu_s}}\, E_{t_0} \cdot \cos\varphi_t = 0$$

$\Downarrow$

$$t_s := \left.\frac{|\vec{E}_{t_0}|}{|\vec{E}_{i_0}|}\right|_{E_\perp} = \frac{2 \cdot \cos\varphi_0 \sqrt{\varepsilon_a \mu_a}}{\sqrt{\varepsilon_a \cdot \mu_a} \cdot \cos\varphi_0 - \frac{\mu_a}{\mu_s} \cdot \sqrt{\mu_s \cdot \varepsilon_s - \mu_a \cdot \varepsilon_a \cdot \sin^2\varphi_0}} \quad ⑤$$

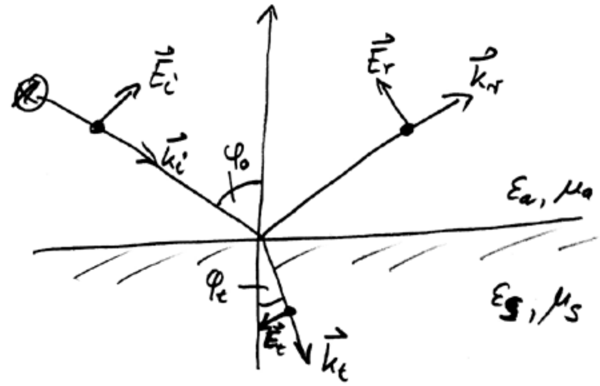
and

$$r_s := \left.\frac{|\vec{E}_{r_0}|}{|\vec{E}_{i_0}|}\right|_{E_\perp} = \frac{\sqrt{\varepsilon_a \cdot \mu_a} \cdot \cos\varphi_0 - \frac{\mu_a}{\mu_s} \cdot \sqrt{\mu_s \cdot \varepsilon_s - \mu_a \cdot \varepsilon_a \sin^2\varphi_0}}{\sqrt{\varepsilon_a \cdot \mu_a}\, \cos\varphi_0 + \frac{\mu_a}{\mu_s}\sqrt{\mu_s \cdot \varepsilon_s - \mu_a\, \varepsilon_a \sin^2\varphi_0}} \quad ⑥$$

$\Rightarrow$

Next, consider the electric field vector ∥ (parallel) to the plane of incidence:

The tangential components $\vec{E}$ and $\vec{H}$ continuous, demand that



(7) $|\vec{E}_{i0}|\cos\varphi_0 - |\vec{E}_{r0}|\cdot\cos\varphi_0 - |\vec{E}_{t0}|\cdot\cos\varphi_t = 0$

(8) $\sqrt{\frac{\varepsilon_a}{\mu_a}}\cdot|\vec{E}_{i0}| + \sqrt{\frac{\varepsilon_a}{\mu}}|\vec{E}_{r0}| - \sqrt{\frac{\varepsilon_s}{\mu_s}}\cdot|\vec{E}_{t0}| = 0$

$$t_p := \left.\frac{|\vec{E}_{t0}|}{|E_{i0}|}\right|_{E_\parallel} = \frac{2\cdot\sqrt{\mu_a\cdot\varepsilon_a}\cdot\sqrt{\mu_s\cdot\varepsilon_s}\cdot\cos\varphi_0}{\mu_a\cdot\varepsilon_s\cos\varphi_0 + \sqrt{\mu_a\cdot\varepsilon_a}\cdot\sqrt{\mu_s\cdot\varepsilon_s - \mu_a\cdot\varepsilon_a\sin^2\varphi_0}} \qquad (9)$$

$$r_p := \left.\frac{|\vec{E}_{r0}|}{|\vec{E}_{i0}|}\right|_{E_\parallel} = \frac{\mu_a\cdot\varepsilon_s\cdot\cos\varphi_0 - \sqrt{\mu_a\cdot\varepsilon_a}\cdot\sqrt{\mu_s\cdot\varepsilon_s - \mu_a\cdot\varepsilon_a\sin^2\varphi_0}}{\mu_a\cdot\varepsilon_s\cos\varphi_0 + \sqrt{\mu_a\cdot\varepsilon_a}\cdot\sqrt{\mu_s\cdot\varepsilon_s - \mu_a\cdot\varepsilon_a\sin^2\varphi_0}} \qquad (10)$$

For normal incidence ($\varphi = 0$) there is no difference between parallel and perpendicular electrical field components:

$$r_s = r_p = \frac{\sqrt{\frac{\mu_a\cdot\varepsilon_s}{\mu_s\cdot\varepsilon_a}} - 1}{\sqrt{\frac{\mu_a\cdot\varepsilon_s}{\mu_s\cdot\varepsilon_a}} + 1}$$

$\mu_a = \mu_s = 1$:

$$r_s = r_p = \frac{\sqrt{\varepsilon_s} - \sqrt{\varepsilon_a}}{\sqrt{\varepsilon_s} + \sqrt{\varepsilon_a}} \underset{\text{no absorption}}{=} \frac{n_s - n_a}{n_s + n_a}$$

and

$$t_p = t_s = \frac{2}{\sqrt{\frac{\mu_a\cdot\varepsilon_s}{\varepsilon_a\cdot\mu_s}} + 1}$$

$\mu_a = \mu_s = 1$

$$t_p = t_s = \frac{2\cdot\sqrt{\varepsilon_a}}{\sqrt{\varepsilon_s} + \sqrt{\varepsilon_a}} \underset{\text{No absorption}}{=} \frac{2\cdot n_a}{n_s + n_a}$$

$\Rightarrow$

Equations (5), (6), (9) and (10) describe any boundary between two mathematically sharp interfaces, where either ε or μ changes.

The coefficients are denoted as <u>Fresnel's Equations.</u>

The Fresnel's coefficients are generally <u>complex</u> values!

The transmission (T) and Reflectivity (R) are given by

$$T_p = \alpha \cdot (t_p \cdot t_p^*) \qquad \alpha := \frac{\sqrt{\varepsilon_s \mu_s - \varepsilon_a \mu_a \sin^2\varphi_0}}{\sqrt{\varepsilon_a \mu_a} \cdot \cos\varphi_0}$$

$$T_s = \alpha \cdot (t_s \cdot t_s^*)$$

and $R_s = r_s \cdot r_s^*$, $R_p = r_p \cdot r_p^*$

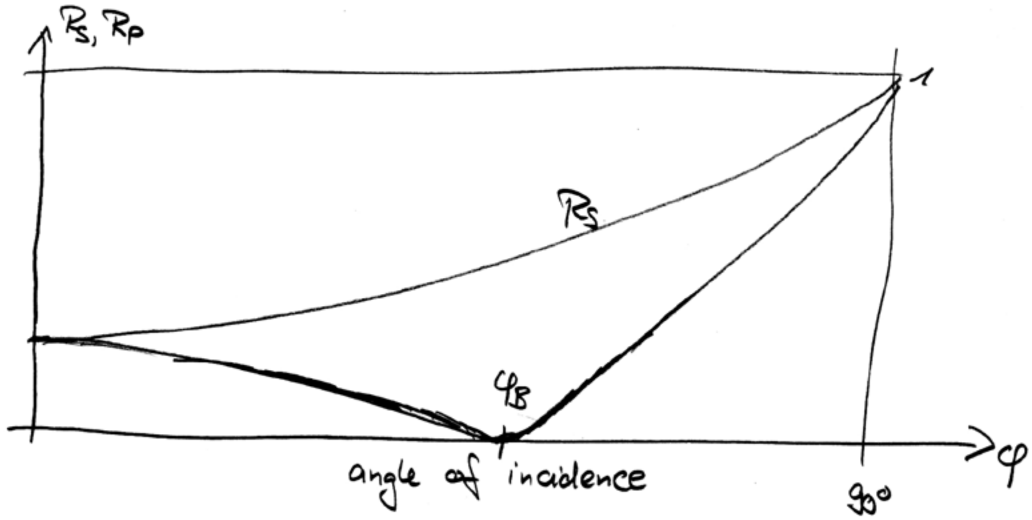
If there is no absorption in the media, we get from the condition that the total energy flow into the area (per time unit) has to be equal the energy flowing outwards (per time unit):

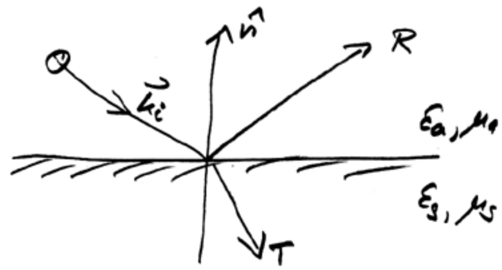
$$J_i \cdot A \cdot \cos\varphi_0 = J_r \cdot A \cdot \cos\varphi_0 + J_t \cdot A \cdot \sqrt{\varepsilon_s \mu_s - \varepsilon_a \mu_a \sin^2\varphi_0} / \sqrt{\varepsilon_s \mu_s}$$

with $R = \frac{J_R}{J_i}$, $T = \frac{J_t \cos\varphi_t}{J_i \cos\varphi_0} = \frac{J_t \cdot \sqrt{\varepsilon_s \mu_s - \varepsilon_a \mu_a \sin^2\varphi_0}}{J_i \cdot \sqrt{\varepsilon_s \mu_s} \cdot \cos\varphi_0}$

and $\underline{R + T = 1}$

with absorption: $\boxed{R + T + A = 1}$ $\Longrightarrow$





<u>Critical angle</u> φ_c:

back to Snell's law: $\sqrt{\varepsilon_a} \cdot \sin\varphi_0 = \sqrt{\varepsilon_s} \cdot \sin\varphi_t$

assume $\sqrt{\varepsilon_a} > \sqrt{\varepsilon_s}$: $\varphi_t = \arcsin\Big[\underbrace{\frac{\sqrt{\varepsilon_a}}{\sqrt{\varepsilon_s}} \cdot \sin\varphi_0} \Big]$

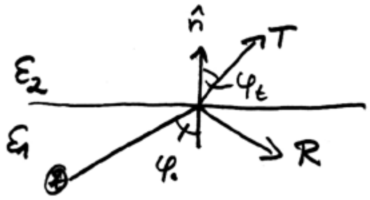
has to be ≤ 1 in order to get real φ_t

$\rightsquigarrow \sin\varphi_0 \leq \frac{\sqrt{\varepsilon_s}}{\sqrt{\varepsilon_a}}$

$\hookrightarrow$ critical angle $\varphi_c := \arcsin\left[\sqrt{\varepsilon_s}/\sqrt{\varepsilon_a}\right]$

$\sqrt{\varepsilon_a} = 1.5$ (Glass) and $\sqrt{\varepsilon_s} = 1 \rightsquigarrow \varphi_c = \underline{\underline{41.8^\circ}}$

<u>Brewster angle</u> φ_B: $\varphi_B := R\big|_{E_\parallel} = 0$ <u>or</u> $\frac{\partial R}{\partial \varphi}\Big|_{E_\parallel} = 0$



Transmitted and reflected rays $\perp$

$\varphi_t + (90 - \varphi_B) = 90$

$\rightsquigarrow$ Snell's law $\sqrt{\varepsilon_1} \sin\varphi_B = \sqrt{\varepsilon_2} \cos\varphi_B \rightsquigarrow \underline{\underline{\varphi_B = \arctan\left(\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\right)}}$