Electromagnetic Waves and Wave Propagation

Maxwell's Equations for the EM field contain the traveling wave solutions, which represent the transport of energy from one point to another!

James C. Maxwell (1873)

Vortices of fields
$$\begin{cases} \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\dot{\vec{B}} \\ \nabla \times \vec{H} = \dot{\vec{D}} + \vec{j} \end{cases}$$

Sources of fields
$$\begin{cases} \nabla \cdot \vec{D} = \rho \\ \nabla \cdot \vec{B} = 0 \end{cases}$$

Materials Equation
$$\begin{cases} \vec{D} = \varepsilon \vec{E} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon_0 \vec{E} + \vec{P} \\ \vec{B} = \mu \vec{H} = \mu_0 \mu_r \vec{H} = \mu_0 \vec{H} + \vec{M} \\ \vec{j} = \sigma \cdot \vec{E} + \frac{\partial \vec{P}}{\partial t} + \nabla \times \vec{M} \end{cases}$$

ε_0: permittivity
μ_0: permeability (free space)
H: magnetic field
E: electric field
B: magnetic Induction
D: electr. Displacement
P: Polarization
M: magnetization
$\vec{j}$: current density
ρ: electr. charge
σ: electrical conductance = conductivity $[A/Vm]$

$$\mu_0 = 4\pi \cdot 10^{-7} \frac{Vs}{Am}, \quad \varepsilon_0 = 8.854 \cdot 10^{-12} \frac{As}{Vm}$$

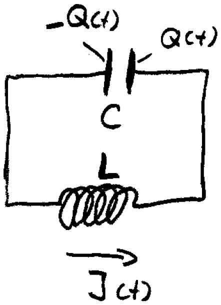
$$\varepsilon_0 \mu_0 = \frac{1}{c^2}; \quad \varepsilon_r \cdot \mu_r \cdot \varepsilon_0 \mu_0 = \varepsilon \cdot \mu = \frac{1}{v^2} = \frac{n^2}{c^2}$$

Generation of electromagnetic waves ??

↳ e.g. Oscillating Hertz dipol

⟹

Thomson Oscillator



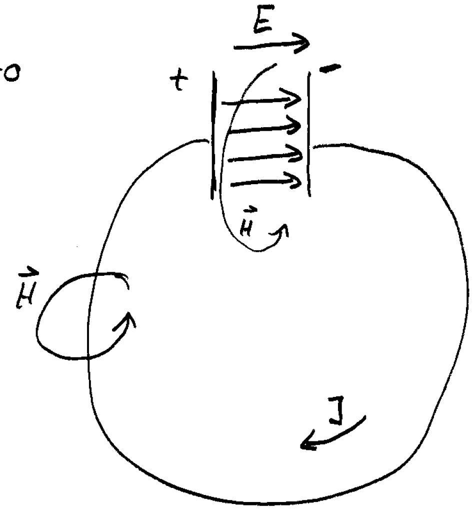
$\sum V_i = 0$ for a closed circuit

$\hookrightarrow \quad \frac{1}{C} Q + L \frac{dJ}{dt} = 0 \, , \quad J = \frac{dQ}{dt}$

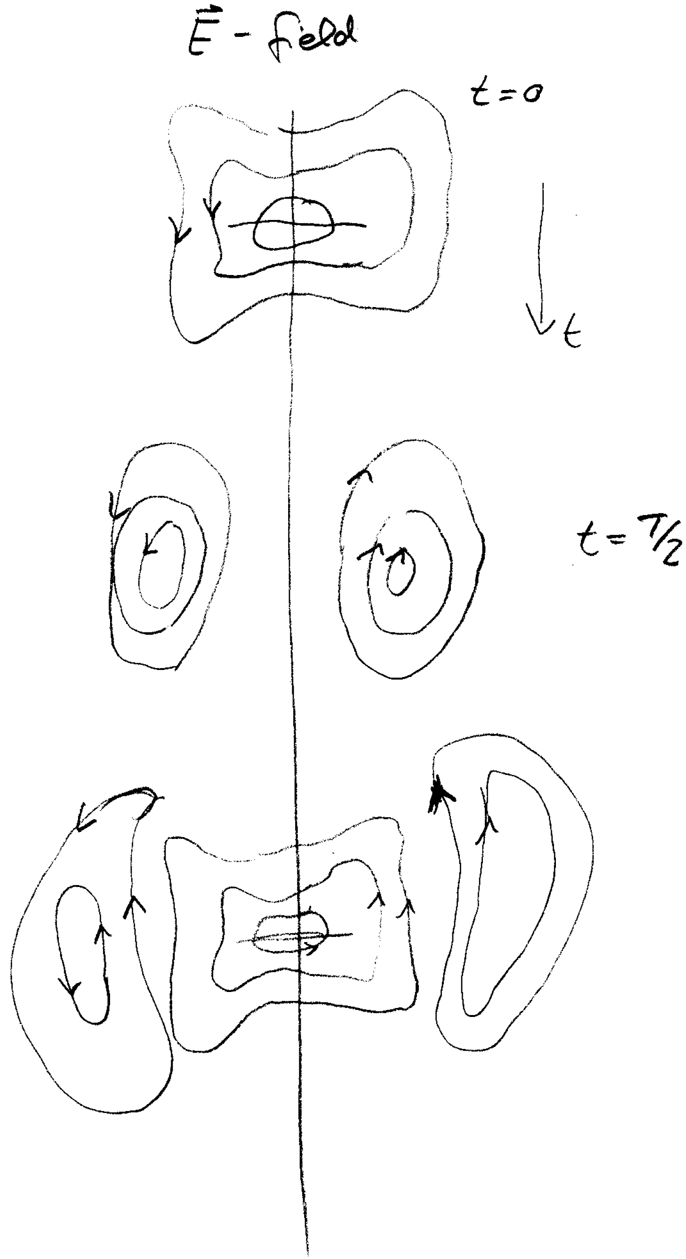
$\hookrightarrow \quad \frac{d^2Q}{dt^2} + \frac{1}{LC} Q = 0 \quad \sim \quad \boxed{\ddot{Q} + \omega_0^2 Q = 0}$

$\omega_0^2 = (L \cdot C)^{-1/2}$

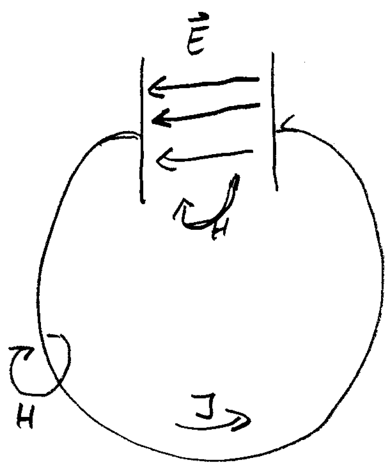
t=0



$\vec{E}$ - field



t = T/2



Heinrich Hertz: wave in space ($\varepsilon_r = \mu_r = 1$)

$$\vec{H} = \frac{1}{c} \nabla \times \frac{\partial \vec{A}}{\partial t} + cc \,, \quad \vec{E} = \frac{\partial^2 \vec{A}}{\partial t^2} + \nabla \varphi$$

with $\varphi = \nabla \cdot \vec{A} \quad \rightsquigarrow \quad \nabla \varphi = \Delta \vec{A}$

$\hookrightarrow$ $\boxed{\Delta \vec{A} - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = 0}$ remember $\varepsilon_0 \cdot \mu_0 = \frac{1}{c^2}$

Use polar coordinates and time-period vector potential $\vec{A}$

$$A(\vec{r},t) = \vec{A}(\vec{r}) \cdot e^{-i\omega t}$$

to get $\frac{\partial^2 A(\vec{r})}{\partial r^2} + \frac{2}{r} \frac{\partial A(\vec{r})}{\partial r} + \frac{\omega^2}{c^2} A(\vec{r}) = 0$

solution: $A(\vec{r},t) = \frac{1}{r} \mu(t)\, e^{i\omega \cdot r/c}$

$\hookrightarrow$ $\vec{H}(\vec{r},t) = \frac{i\omega}{c \cdot r} \left[\frac{1}{r} + \frac{i\omega}{c} \right] e^{i\omega (r/c - t)} \cdot \left[\vec{\mu}_0 \times \vec{r}_0 \right]$

$\mu(t) = \mu_0\, e^{-i\omega t}$, remember $\frac{\omega}{c} = \frac{2\pi}{\lambda}$

$\hookrightarrow$ two limiting cases:

(i) $r \ll \lambda$: near field

(ii) $r \gg \lambda$: far field

(i) near field $r \ll \lambda$

$$\frac{d\mu(t)}{dt} = |\vec{j}|\, d\vec{e} \;\rightarrow\; \vec{\mu} = -q\, d\vec{e} \;:\; \text{a dipol}$$

$\Longrightarrow$

(ii) far field $r \gg \lambda$

$$\rightsquigarrow \vec{H}(\vec{r},t) = \frac{1}{c^2 r}\, e^{\frac{i\omega r}{c}} \left[\frac{\partial^2 \mu}{\partial t^2} \times \vec{r}_0 \right] + c.c.$$

↳ vanishing amplitude at poles ($\varphi = 0$)

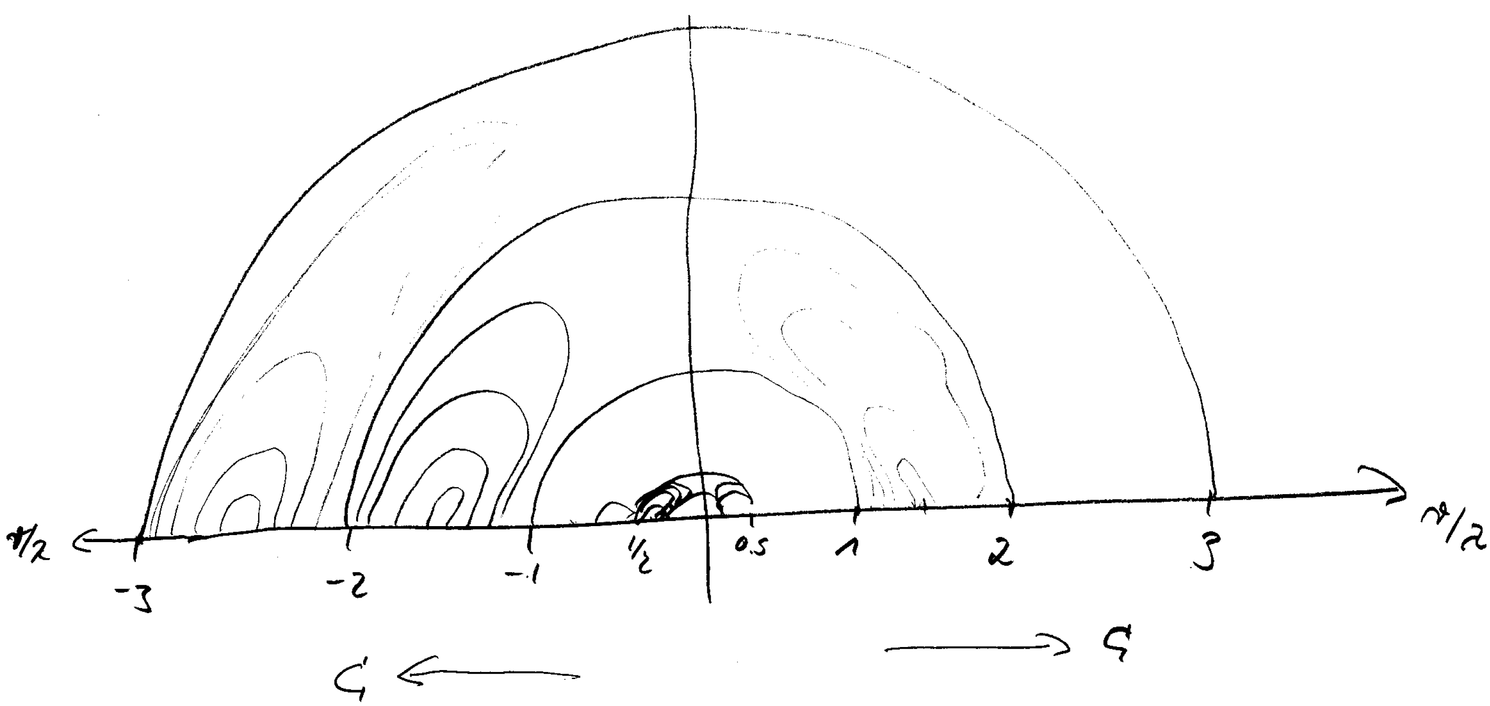
$$|\vec{H}| = |\vec{E}| = \frac{\omega}{c^2}\, \frac{1}{r}\, |\mu_0| \sin\varphi \cdot \cos\left(\frac{\omega_0 r}{c} - \omega t \right)$$

Poynting Vector $S = \vec{E} \times \vec{H}$

$$|\vec{S}| = \frac{\omega^4}{4\pi c^3 r^2} |\mu_0|^2 \sin^2\varphi \cos^2\left(\frac{\omega_0 r}{c} - \omega t \right)$$

time average $\overline{|\vec{S}|} = \frac{\omega^4}{8\pi c^3 r^2} |\mu_0|^2 \sin^2\varphi$

total emitted energy $\oint |\vec{S}|\, ds = \frac{\omega^4}{\pi c^3} |\vec{\mu}_0|^2$



EM - wave propagation ; undamped plane waves

(1) Consider an unbounded (semi-infinite) media, isotrop and non-conducting.

↳ In absence of sources:

$$\boxed{\begin{array}{ll} \nabla\cdot\vec{B}=0 \;; & \nabla\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0 \\ \nabla\cdot\vec{D}=0 \;; & \nabla\times\vec{H}-\partial\vec{D}/\partial t=0 \end{array}}$$ — Maxwell Equations

Next, consider harmonic solutions : $\vec{E},\vec{B}\sim e^{i\omega t}$

$$\vec{E}(\vec{r},t)=\vec{E}(\omega,\vec{r})\cdot e^{-i\omega t}$$ ($\vec{H},\vec{B}$ and $\vec{D}$ similar!)

↳ $$\boxed{\begin{array}{ll} \nabla\cdot\vec{B}=0 \;; & \vec{\nabla}\times\vec{E}-i\omega\vec{B}=0 \\ \nabla\cdot\vec{D}=0 \;; & \nabla\times\vec{H}+i\omega\vec{D}=0 \end{array}}$$ Maxwell's Equation for harmonic fields

For a uniform, isotropic and linear medium

$$\vec{D}=\varepsilon\vec{E}$$
$$\vec{B}=\mu\vec{H}$$

→ ε,μ are complex functions of $\underline{\omega}$!

Let first assume that ε and μ are real (No losses!)

↳ (a) $\nabla\times\vec{E}-i\omega\vec{B}=0$ and **(b)** $\nabla\times\vec{B}+i\omega\mu\varepsilon\vec{E}=0$

↳ $\vec{E}=\frac{-1}{i\omega\mu\varepsilon}\nabla\times\vec{B}$

$$-\nabla\times\left(\frac{\nabla\times\vec{B}}{i\omega\mu\varepsilon}\right)-i\omega\vec{B}=0$$

with $\nabla\times(\nabla\times\vec{B})=\nabla(\underbrace{\nabla\cdot\vec{B}}_{=0})-\nabla^2\vec{B}=-\Delta\vec{B}\implies$

$$\rightsquigarrow \frac{1}{i\omega\mu\varepsilon}\Delta\vec{B} - i\omega\vec{B} = 0 \quad \rightarrow \quad \boxed{\Delta\vec{B} + \mu\varepsilon\omega^2\vec{B} = 0}$$

from (a) $\vec{B} = \frac{1}{i\omega}$ substitute in (b)

$$\nabla\times\left(\frac{1}{i\omega}\nabla\times\vec{E}\right) + i\omega\mu\varepsilon\vec{E} = 0$$

$$-\frac{\nabla^2\vec{E}}{i\omega} + i\omega\mu\varepsilon\vec{E} = 0$$

$$\hookrightarrow \boxed{\Delta\vec{E} - \omega^2\mu\varepsilon\vec{E} = 0}$$

These are the Helmholtz equations

Trail solutions: $\psi = A_0 e^{i\vec{k}\cdot\vec{r} - i\omega t} = \psi(\omega,\vec{r})\cdot e^{-i\omega t}$

DE: $\Delta\psi - \omega^2\mu\varepsilon\psi = -k^2 A_0 e^{i\vec{k}\vec{r} - i\omega t} + \omega^2\mu\varepsilon e^{i\vec{k}\vec{r} - i\omega t} = 0$

$$\hookrightarrow k^2 = \omega^2\mu\varepsilon$$

$$k = \omega\sqrt{\mu\varepsilon} = \omega\cdot v$$

Phase velocity $v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n}$, $n = \sqrt{\frac{\mu\cdot\varepsilon}{\mu_0\varepsilon_0}} \geq 1$

n - index of refraction; $n = n(\omega)$

basic solution in 1D: $u(x,t) = a\, e^{ikx - i\omega t} + b\cdot e^{-ikx - i\omega t}$

$$= a\, e^{ik(x - v\cdot t)} + b\, e^{-ik(x + v\cdot t)}$$

general solution:

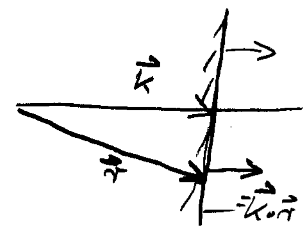
$\hookrightarrow u(x,t) = f(x - v\cdot t) + g(x + v\cdot t)$ f, g - arbitrary

$\Longrightarrow$

Consider an EM plane wave of frequency ω and wavevector $\vec{k} = k \cdot \hat{n}$

plane wave:

$\vec{k} \cdot \vec{r} =$ Constant



$$\rightsquigarrow \quad \vec{E}(\vec{r},t) = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$$

$$\text{with} \quad \omega = v \cdot |\vec{k}| = v \cdot k \cdot \underbrace{|\hat{n}|}_{=1} = v \cdot k$$

From Poisson equation

$$\nabla \cdot \vec{E} = -\vec{k} \cdot \vec{E}_0 \sin(\vec{k}\cdot\vec{r} - \omega t) = 0$$

$$\rightsquigarrow \quad \vec{k} \perp \vec{E}_0$$

and analogous $\vec{k} \perp \vec{H}_0$

↳ EM-waves are transversal waves and E_0 & H_0 define the polarization of the wave.

Using the Faraday law

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad \rightsquigarrow \quad \vec{H} = \int \frac{1}{\mu} \nabla \times \vec{E} \, dt$$

$$\boxed{\begin{array}{l} \text{if } \vec{X} = \vec{X}_0 \, e^{i\vec{k}\cdot\vec{r}} \\ \nabla \cdot \vec{X} = \vec{k} \cdot \vec{X} \\ \nabla \times \vec{X} = \vec{k} \times \vec{X} \end{array}}$$

$$\hookrightarrow \quad \vec{H} = \frac{1}{\mu} \int \vec{k} \times \vec{E}_0 \sin(\vec{k}\cdot\vec{r} - \omega t)$$

$$= \frac{1}{\mu \cdot \omega} \vec{k} \times \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$$

$$= \vec{H}_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$$

$$\hookrightarrow \quad H_0 = \frac{1}{\mu \omega} \vec{k} \times \vec{E}_0 \quad \rightsquigarrow \quad \vec{H}_0 \perp \vec{E}_0$$

↳ E_0, H_0 and $\vec{k}$ form right-turn system! ⟹

Knowing H_0 and E_0, we can calculate the energy flux density $\vec{S}$ (Poynting Vector) and the energy density $u(\vec{r},t)$ of an EM-field:

$$S := \vec{E} \times \vec{H} = \frac{E_0 \times (\vec{k} \times \vec{E}_0)}{\mu \cdot \omega} \cos^2(\vec{k} \cdot \vec{r} - \omega t)$$

$$= \frac{\vec{k} \cdot \vec{E}_0^2}{\mu \omega} \cos^2(\vec{k} \cdot \vec{r} - \omega t)$$

$$\hookrightarrow \vec{S} \upuparrows \vec{k}$$

<u>Energy density:</u> $$u(\vec{r},t) = \frac{1}{2} (\vec{E} \cdot \vec{D} + \vec{H} \cdot \vec{B}) = \frac{1}{2} \varepsilon \vec{E}^2 + \frac{1}{2} \mu \vec{H}^2$$

$$= \frac{1}{2} \varepsilon E_0^2 \cos^2(\vec{k} \cdot \vec{r} - \omega t) + \frac{1}{2} \mu \frac{(\vec{k} \times E_0^2)}{\mu^2 \omega^2} \cos^2(\vec{k} \cdot \vec{r} - \omega t)$$

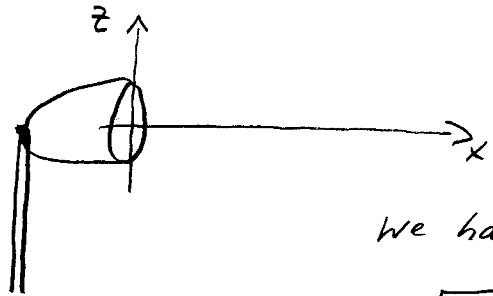
Since $\frac{1}{\mu \omega^2} (\vec{k} \times \vec{E}_0)^2 = \frac{1}{\mu \omega^2} (\vec{k} \times E_0)^2 = \frac{1}{\mu \omega^2} \vec{k}^2 \vec{E}_0^2 = \frac{1}{\mu v^2} \vec{E}_0^2 = \varepsilon \vec{E}_0^2$

we get $$u(r,t) = \varepsilon_0 \vec{E}_0^2 \cos^2(\vec{k} \cdot \vec{r} - \omega t)$$

Note: not only plane EM-wave but any arbitrary wave package propagate with the velocity $\vec{v}$ in space/medium!

$\Longrightarrow$

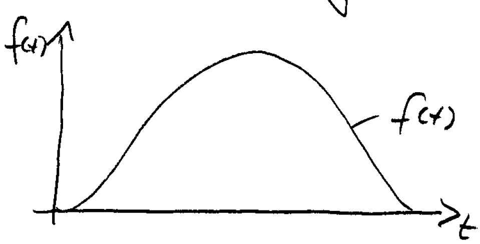
Example: light flash from flash lamp



We have to solve the wave DE

$$\boxed{\Delta E_z(x,t) - \frac{1}{v^2}\ddot{\vec{E}}_z(x,t) = 0}$$

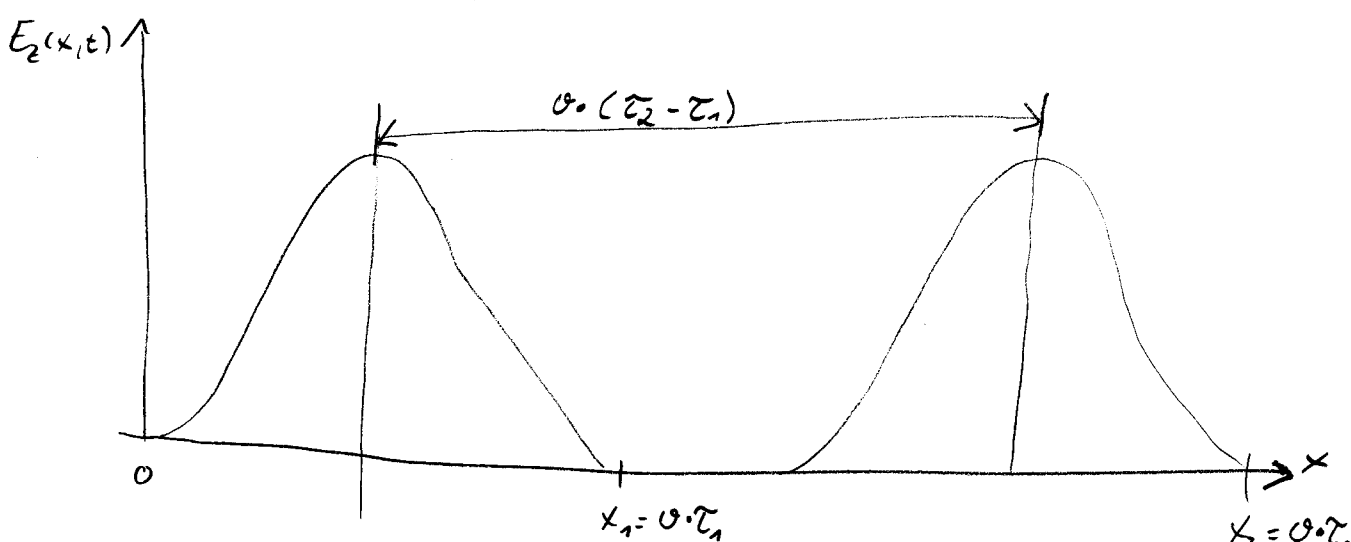
with the boundary value $E_z(0,t) = f(t)$



$f(t) \sim \sin^2(\omega t)$, $\delta(t_0 - t)$ . . .

A solution of the wave equation which satisfies the boundary value is given by

$$E_z(x,t) = f\left(t - x\frac{k}{\omega}\right) = f\left(t - \frac{x}{v}\right)$$



EM- wave propagation: damped waves in conducting media.

Maxwell equations:

$$\nabla \times \vec{E} = -\dot{\vec{B}} \quad / \times \nabla$$

(1) $\quad \hookrightarrow \quad \nabla \times (\nabla \times \vec{E}) = -\mu (\nabla \times \dot{\vec{H}})$

and $\quad \nabla \times \vec{H} = \varepsilon_0 \dot{\vec{E}} + \sigma \vec{E}$

(2) $\quad \hookrightarrow \quad \nabla \times \dot{\vec{H}} = \varepsilon \ddot{\vec{E}} + \sigma \dot{\vec{E}}$

Substitute (2) in (1)

$$\nabla \times (\nabla \times \vec{E}) = -\mu [\varepsilon \ddot{\vec{E}} + \sigma \dot{\vec{E}}]$$

$$\downarrow$$

$$\nabla(\nabla \cdot \vec{E}) - \Delta \vec{E} = -\mu\varepsilon \ddot{\vec{E}} - \mu\sigma \dot{\vec{E}}$$

Assume No electr. sources $\quad \nabla \cdot \vec{E} = 0$

$\hookrightarrow$

$$\boxed{\begin{array}{l} \Delta E - \mu\sigma \dot{\vec{E}} - \mu\varepsilon \ddot{\vec{E}} = 0 \\ \Delta \vec{H} - \mu\sigma \dot{\vec{E}} - \mu\varepsilon \ddot{\vec{E}} = 0 \end{array}}$$

<u>Telegraph equations</u>

analogous:

with $\mu\varepsilon = \frac{1}{v^2} = \mu_r \varepsilon_r \mu_0 \varepsilon_0 = \mu_r \varepsilon_r \frac{1}{c^2} = \frac{n^2}{c^2}$

$$\hookrightarrow \quad \Delta E - \mu\sigma \dot{\vec{E}} - \frac{1}{v^2} \ddot{\vec{E}} = 0$$

Consider a special solution for a plane EM wave propagating in x-direction. $\rightarrow \vec{E}$ and $\vec{H}$ only depend on x, t - not on y and z!

$\Longrightarrow$

$\rightsquigarrow$ the Telegraph DE:

$$\boxed{\frac{\partial^2 E}{\partial x^2} - \mu\sigma \frac{\partial E}{\partial t} - \frac{1}{v^2}\frac{\partial^2 E}{\partial t^2} = 0}$$

From the Poisson equation

$$\nabla \cdot \vec{E} = 0 \quad \rightsquigarrow \quad \frac{\partial E_x}{\partial x} = 0$$

Since we only consider spatial varible fields, we can set $\underline{E_x = 0}$!

For the y- and z-components of the wave $\vec{E}$, we use the separation trail solution:

$$E_\nu(x,t) = f_\nu(x)\, e^{i\omega t} \quad , \quad (\nu = y, z)$$

$$\hookrightarrow \quad f_\nu''(x) - i\omega\mu\sigma f_\nu + \frac{1}{v^2}\omega^2 f_\nu = 0$$

$$\hookrightarrow \quad \boxed{f_\nu'' + \left(\frac{\omega^2}{v^2} - i\omega\mu\sigma\right) f_\nu = 0}$$

The solution of this DE is

$$f_\nu(x) = f_\nu(0)\, \exp\left(\pm i x \sqrt{\frac{\omega^2}{v^2} - i\mu\sigma\omega}\right)$$

Since the propagation is in positive x-direction, the positive sign disappears and we get for the complete solution

$$E_\nu(x,t) = f_\nu(x) \cdot e^{i\omega t} = E_{0\nu} \cdot \exp\left[i\omega t - i x \frac{\omega}{c}\sqrt{\underbrace{\frac{c^2}{\omega^2}}_{=n} - \underbrace{i\frac{\mu\sigma c^2}{\omega}}_{k}}\right]$$

$\Rightarrow$

next we define the complex refractive index $\hat{n}$

with $\boxed{\hat{n}^2 = \frac{c^2}{v^2} - i\,\frac{\mu\sigma c^2}{\omega}}$ $\Rightarrow$ $\hat{n} = n - ik$

$\hookrightarrow$ $E_y(x,t) = E_{oy} \exp[i\omega t - ix\frac{\omega}{c}\cdot\hat{n}]$ – damped wave propagation

analogous to $E_y(x,t) = E_{oy}\exp[i\omega t - ix\frac{\omega}{c}\cdot n]$ – solution of undamped wave propagation

undamped wave propagation	damped wave propag.
$n = \frac{c}{v} = \sqrt{\mu_r\varepsilon_r}$	$\hat{n} = n - ik = \sqrt{\frac{c^2}{v^2} - i\frac{\mu\sigma c^2}{\omega}}$
	$k = \frac{\mu\sigma c^2}{\omega}$ = <u>extinction coefficient</u>

Introduce complex dielectric function $\hat{\varepsilon} = \varepsilon_1 - i\varepsilon_2$

with $\hat{\varepsilon} = \hat{n}^2 = (n-ik)^2 = n^2 - k^2 - i\cdot 2nk = \frac{c^2}{v^2} - i\cdot\frac{\mu\sigma c^2}{\omega}$

$\hookrightarrow$
$$\boxed{\begin{aligned} Re\{\hat{\varepsilon}\} &= \varepsilon_1 = n^2 - k^2 = \frac{c^2}{v^2} = \varepsilon_r\cdot\mu_r \\ Im\{\hat{\varepsilon}\} &= \varepsilon_2 = 2nk = \frac{\mu_r\sigma}{\varepsilon_0\cdot\omega} \end{aligned}}$$

reverse: $n = \sqrt{(\|\hat{\varepsilon}\| + \varepsilon_1)/2}$

$k = \sqrt{(\|\hat{\varepsilon}\| - \varepsilon_1)/2}$

~> Solution of Telegraph equation:

$$\vec{E}_y(x,t) = E_{oy} \exp[i\omega t - ix \tfrac{\omega}{c} \cdot \hat{n}]$$

$$= E_{oy} \exp[i\omega(t - \tfrac{n}{c} \cdot x)] \cdot \exp[-x \cdot \tfrac{\omega}{c} \cdot k]$$

The factor $\exp[-x\frac{\omega}{c}k]$ is the damping factor of the wave!
(k – extinction coefficient)

Since light according to the Maxwell theory can be treated as an electromagnetic wave, it has to be feasible to express the materials property: refractive index and absorption coefficient α through the electromagnetic constants ε, μ and σ.

(A) <u>Link to absorption coefficient α</u>

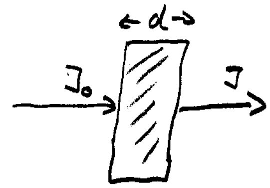
First, lets define the term intensity, J: it is the time-average of the EM-energy density

$$J = \langle u \rangle = \frac{1}{T}\int_{t_0}^{t_1} u \, dt \quad , \quad \langle u \rangle = \langle \varepsilon \vec{E}^2 \rangle$$

$$= \langle \varepsilon_0 (E_{oy}^2 + E_{oz}^2) \cdot \exp[-2x\tfrac{\omega}{c}k] \cdot \cos^2[\omega \cdot (t - \tfrac{n}{c}x)] \rangle$$

$$= \frac{1}{2} \varepsilon_0 \vec{E}_0^2 e^{-x \cdot \frac{2\omega k}{c}}$$

→ Now, lets assume an EM-wave of intensity J_0 goes through a plate of thickness d:



Experimental, the exiting intensity is linked to the entering wave via $\boxed{\frac{J}{J_0} = e^{-\alpha \cdot d}}$

α: Absorption coefficient
d: thickness

If we now use for J and J_0 the time-average energy densities, we obtain:

$$\frac{\frac{1}{2}\varepsilon_0 \vec{E}_0^2 \cdot \exp\left[-d \frac{2\omega k}{c}\right]}{\frac{1}{2}\varepsilon_0 \vec{E}_0^2} = \exp\left[-d \frac{2\omega k}{c}\right] \stackrel{!}{=} \exp(-\alpha d)$$

$$\hookrightarrow \quad \alpha = \frac{2\omega}{c} \cdot k$$

Using the complex values for $\hat{n}$ and $\hat{\varepsilon}$, we get

$$\hat{n} = n - ik = n - i\frac{\alpha \cdot c}{2\omega}$$

and $$\mathrm{Re}\{\hat{\varepsilon}\} = \varepsilon_1 = n^2 - \frac{\alpha^2 c^2}{4\omega^2} = \varepsilon_r \mu_r$$

$$\mathrm{Im}\{\hat{\varepsilon}\} = \varepsilon_2 = \frac{\alpha \cdot c \cdot n}{\omega} = \frac{\mu_r \cdot \sigma}{\varepsilon_0 \cdot \omega}$$

→ next: link to refractive index n → Dispersion relations