Physics 8100 - Electromagnetic Theory I

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Useful and useless relations

Explicit Forms of Vector Operations

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\Psi = e_1 \frac{\partial \Psi}{\partial x_1} + e_2 \frac{\partial \Psi}{\partial x_2} + e_3 \frac{\partial \Psi}{\partial x_3}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla \times \mathbf{A} = e_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + e_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + e_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\Psi = \Delta\Psi = \frac{\partial^2 \Psi}{\partial x_1^2} + \frac{\partial^2 \Psi}{\partial x_2^2} + \frac{\partial^2 \Psi}{\partial x_3^2}$$

Cylindrical (ρ, Φ, z)

$$\nabla\Psi = e_1 \frac{\partial \Psi}{\partial \rho} + e_2 \frac{1}{\rho} \cdot \frac{\partial \Psi}{\partial \Phi} + e_3 \frac{\partial \Psi}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \cdot \frac{\partial}{\partial \rho}(\rho \cdot A_1) + \frac{1}{\rho} \cdot \frac{\partial A_2}{\partial \Phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla \times \mathbf{A} = e_1\left(\frac{1}{\rho} \cdot \frac{\partial A_3}{\partial \Phi} - \frac{\partial A_2}{\partial z}\right) + e_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial \rho}\right) + e_3 \frac{1}{\rho} \cdot \left(\frac{\partial}{\partial \rho}(\rho \cdot A_2) - \frac{\partial A_1}{\partial \Phi}\right)$$

$$\nabla^2\Psi = \Delta\Psi = \frac{1}{\rho} \cdot \frac{\partial}{\partial \rho}\left(\rho \cdot \frac{\partial \Psi}{\partial \rho}\right) + \frac{1}{\rho^2} \cdot \frac{\partial^2 \Psi}{\partial \Phi^2} + \frac{\partial^2 \Psi}{\partial z^2}$$

Spherical (ρ, θ, Φ)

$$\nabla\Psi = e_1 \cdot \frac{\partial \Psi}{\partial r} + e_2 \cdot \frac{1}{r} \cdot \frac{\partial \Psi}{\partial \theta} + e_3 \cdot \frac{1}{r \cdot \sin\theta} \cdot \frac{\partial \Psi}{\partial \Phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \cdot \frac{\partial}{\partial r}\left(r^2 A_1\right) + \frac{1}{r \cdot \sin\theta} \cdot \frac{\partial}{\partial \theta}(\sin\theta \cdot A_2) + \frac{1}{r \cdot \sin\theta} \cdot \frac{\partial A_3}{\partial \Phi}$$

$$\nabla \times \mathbf{A} = e_1 \cdot \frac{1}{r \cdot \sin\theta} \cdot \left[\frac{\partial}{\partial \theta}(\sin\theta \cdot A_3) - \frac{\partial A_2}{\partial \Phi}\right] + e_2 \cdot \left[\frac{1}{r \cdot \sin\theta} \cdot \frac{\partial A_1}{\partial \Phi} - \frac{1}{r} \cdot \frac{\partial}{\partial r}(r \cdot A_3)\right] + e_3 \frac{1}{r} \cdot \left(\frac{\partial}{\partial r}(r \cdot A_2) - \frac{\partial A_1}{\partial \theta}\right)$$

$$\nabla^2\Psi = \frac{1}{r^2} \cdot \frac{\partial}{\partial r}\left(r^2 \cdot \frac{\partial \Psi}{\partial \rho}\right) + \frac{1}{r^2 \cdot \sin\theta} \cdot \frac{\partial}{\partial \theta}\left(\sin\theta \cdot \frac{\partial \Psi}{\partial \theta}\right) + \frac{1}{r^2 \cdot \sin^2\theta} \frac{\partial^2 \Psi}{\partial \Phi^2}$$

$$\left\{ \text{note that } \frac{1}{r^2} \cdot \frac{\partial}{\partial r}\left(r^2 \cdot \frac{\partial \Psi}{\partial \rho}\right) \equiv \frac{1}{r} \cdot \frac{\partial^2}{\partial r^2}(r \cdot \Psi) \right\}$$

Vector Formulas

$$\vec{A} \bullet (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \bullet \vec{C}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} \bullet (\vec{A} \bullet \vec{C}) - \vec{C} \bullet (\vec{A} \bullet \vec{B})$$

$$(\vec{A} \times \vec{B}) \bullet (\vec{C} \times \vec{D}) = (\vec{A} \bullet \vec{C}) \cdot (\vec{B} \bullet \vec{D}) - (\vec{A} \bullet \vec{D}) \cdot (\vec{B} \bullet \vec{C})$$

$$\nabla \times \nabla \Psi = 0$$

$$\nabla \bullet (\nabla \times \vec{A}) = 0$$

$$\nabla \times \left(\nabla \times \vec{A}\right) = \nabla \left(\nabla \bullet \vec{A}\right) - \nabla^2 \vec{A}$$

$$\nabla \bullet \left(\Psi \cdot \vec{A}\right) = \vec{A} \cdot \nabla \Psi + \Psi \cdot \left(\nabla \bullet \vec{A}\right)$$

$$\nabla\left(\vec{A} \bullet \vec{B}\right) = \left(\vec{A} \bullet \nabla\right)\vec{B} + \left(\vec{B} \bullet \nabla\right)\vec{A} + \vec{A} \times \left(\nabla \times \vec{B}\right) + \vec{B} \times \left(\nabla \times \vec{A}\right)$$

$$\nabla \bullet \left(\vec{A} \times \vec{B}\right) = \vec{B} \bullet \left(\nabla \times \vec{A}\right) - \vec{A} \bullet \left(\nabla \times \vec{B}\right)$$

$$\nabla \times \left(\vec{A} \times \vec{B}\right) = \vec{A}\left(\nabla \bullet \vec{B}\right) - \vec{B}\left(\nabla \bullet \vec{A}\right) + \left(\vec{B} \bullet \nabla\right)\vec{A} - \left(\vec{A} \bullet \nabla\right)\vec{B}$$

If $\vec{x}$ is the coordinate of a point with respect to some origin, with magnitude $r = |\vec{x}|$, $\vec{n} = \vec{x}/r$ is a unit radial vector, and f(r) is a well-behaved function of r, then

$$\nabla \bullet \vec{x} = 3; \qquad \nabla \times \vec{x} = 0$$

$$\nabla \bullet [\, \vec{n} \cdot f(r) \,] = \frac{2}{r} \cdot f + \frac{\partial f}{\partial r}; \qquad \nabla \times [\, \vec{n} \cdot f(r) \,] = 0$$

$$(\vec{a} \bullet \nabla)\vec{n} \cdot f(r) = \frac{f(r)}{r} \cdot \left[\vec{a} - \vec{n}(\vec{a} \bullet \vec{n})\,\right] + \vec{n}(\vec{a} \bullet \vec{n})\frac{\partial f}{\partial r}$$

$$\nabla(\vec{a} \bullet \vec{x}) = \vec{a} + \vec{x}(\nabla \bullet \vec{a}) + i\,(L \times \vec{a});$$

$$\text{where } L = \frac{1}{i}(\vec{x} \times \vec{a}) \text{ is the angular - momentum operator}$$

Theorems from Vector Calculus

In the following Φ, Ψ, and $\vec{A}$ are well-behaved scalar or vector functions, V is a three-dimensional volume with volume element $d^3\vec{x}$, S is a closed two-dimensional surface bounding V, with the area element $d\vec{a}$ and unit outward normal $\vec{n}$ at $d\vec{a}$.

$$\int_v \nabla \bullet \vec{A}\, d^3\vec{x} = \int_S \vec{A} \bullet \vec{n}\, d\vec{a}$$ (Divergence theorem)

$$\int_v \nabla \psi\, d^3\vec{x} = \int_S \psi\, \vec{n}\, d\vec{a}$$

$$\int_v \nabla \times \vec{A}\, d^3\vec{x} = \int_S \vec{n} \times \vec{A}\, d\vec{a}$$

$$\int_v \left(\phi \nabla^2 \psi + \nabla \phi \bullet \nabla \psi\right) d^3\vec{x} = \int_S \phi\, \vec{n} \bullet \nabla \psi\, d\vec{a}$$ (Green’s first identity)

$$\int_v \left(\phi \nabla^2 \psi + \psi \nabla^2 \phi\right) d^3\vec{x} = \int_S (\, \phi \nabla \psi - \psi \nabla \phi \,) \bullet \vec{n}\, d\vec{a}$$ (Green’s theorem)

In the following S is a open surface and C is the contour bounding it, with line element $d\vec{l}$. The normal $\vec{n}$ to S is defined by the right-hand screw rule in relation to the sense of the line integral around C.

$$\int_s \left(\nabla \times \vec{A}\right) \bullet \vec{n} \cdot d\vec{a} = \int_C \vec{A} \cdot d\vec{l}$$ (Stokes’s theorem)

$$\int_s (\vec{n} \times \nabla \psi)\, d\vec{a} = \int_C \psi \cdot d\vec{l}$$

Multipole expansion:

Cartezian coordinates:

$q = \int \rho(\mathbf{x}')\, \mathbf{d^3x'}$

$\mathbf{p} = \int \mathbf{x}'\rho(\mathbf{x}')\, \mathbf{d^3x'}$

$Q_{ij} = \int (3x'_i x'_j - r'^2\delta_{ij})\, \rho(\mathbf{x}')\, \mathbf{d^3x'}$

$\varphi(\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{4\pi\varepsilon_0}}\left[\frac{\mathbf{q}}{\mathbf{r}} + \frac{\mathbf{p\cdot x}}{\mathbf{r^3}} + \frac{\mathbf{1}}{\mathbf{2}}\sum_{\mathbf{i,j}} \mathbf{Q_{ij}}\frac{\mathbf{x_i x_j}}{\mathbf{r^5}} + ...\right]$

Spherical coordinates:

$q_{lm} = \int Y^*_{lm}(\theta', \varphi')\, r'^l\, \rho(\mathbf{x}')\mathbf{d^3\,x'}$

$q_{l,-m} = (-1)^m q^*_{lm}$

$\varphi(\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{4\pi\varepsilon_0}}\sum_{\mathbf{l=0}}^{\infty}\sum_{\mathbf{m=-l}}^{\mathbf{l}}\frac{\mathbf{4\pi}}{\mathbf{2l+1}}\mathbf{q_{lm}}\frac{\mathbf{Y_{lm}(\theta,\varphi)}}{\mathbf{r^{l+1}}}$

$Y_{00} = \frac{1}{\sqrt{4\pi}}$

$Y_{11} = -\sqrt{\frac{3}{4\pi}}\sin\theta\, e^{i\varphi}, \qquad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$

$Y_{lm}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)\, e^{im\varphi}$

$\int_{-1}^{1} P_{l'}^m(x)P_l^m(x)dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{l'l}$

Legendre polynomials:

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}(3x^2-1) \\
P_3(x) &= \frac{1}{2}(5x^3-3x) \\
P_4(x) &= \frac{1}{8}(35x^4-30x^2+3)
\end{aligned}
$$

Expansion in terms of Legendre polynomials:

$f(x) = \sum_{l=0}^{\infty} A_l P_l(x)$

$A_l = \frac{2l+1}{2}\int_{-1}^{1} f(x)\, P_l(x)\, dx$

Boundary-value problem with azimuthal symmetry:

$\varphi(r,\theta) = \sum_{l=0}^{\infty}[A_l r^l + B_l r^{-(l+1)}]P_l(\cos\theta)$

A_l, B_l - from boundary conditions

If $V(\theta)$ - potential on the surface of a sphere of radius a, then for potential inside the sphere:

$B_l = 0$

$V(\theta) = \sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta), \qquad A_l = \frac{2l+1}{2a^l}\int_0^{\pi} V(\theta)P_l(\cos\theta)\sin\theta\, d\theta$