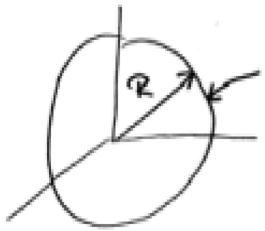
Solutions for Homework # 8

Problem 1: *(20 points)*

A line charge density $\sigma(\vartheta) = K \cdot \cos(\vartheta)$ is glued over the surface of a spherical shell of radius R, where K is a constant. Find the resulting potential inside and outside the sphere.



$\sigma(\theta) = K \cdot \cos(\theta)$ (glued charge density)

Solution for inside the sphere

$$\boxed{V(r,\theta) = \sum_\ell A_\ell \, r^\ell P_\ell(\cos\theta)} \qquad r \leq R$$

for outside: $$V(r,\theta) = \sum_\ell B_\ell \, r^{-(\ell+1)} P_\ell(\cos\theta) \qquad r \geq R$$

1.) Potential at surface is continuous $r = R$

$$\sum_\ell A_\ell R^\ell P_\ell(\cos\theta) = \sum_\ell B_\ell R^{-(\ell+1)} P_\ell(\cos\theta) \qquad \Big/ \cdot \int_0^\pi P_{\ell'}(\cos\theta)\, d\theta$$

$$\sum_\ell A_\ell R^\ell \int_0^\pi P_\ell(\cos\theta) \cdot P_{\ell'}(\cos\theta)\, d\theta = \sum_\ell B_\ell \cdot R^{-(\ell+1)} \cdot \underbrace{\int_0^\pi P_\ell(\cos\theta) \cdot P_{\ell'}(\cos\theta)\, d\theta}_{\delta(\ell-\ell') \to \ell=\ell'}$$

$$\leadsto A_\ell R^\ell \cdot \frac{2}{2\ell+1} = B_\ell \, R^{-(\ell+1)} \cdot \frac{2}{2\ell+1}$$

$$\leadsto B_\ell = A_\ell R^{2\ell+1}$$

2.) $$\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{inside}}{\partial r}\Big|_{r=R} = 4\pi\sigma$$

$$\hookrightarrow \left[\sum_\ell B_\ell (-\ell-1)\, r^{-(\ell+2)} P_\ell(\cos\theta) - \sum_\ell A_\ell \cdot \ell \cdot r^{\ell+1} P_\ell(\cos\theta)\right]_{r=R} = -4\pi\sigma$$

and $B_\ell = A_\ell R^{2\ell+1}$

$$\hookrightarrow \sum_\ell \left[A_\ell R^{2\ell+1} \cdot (-\ell-1)\, R^{-(\ell+2)} - A_\ell \cdot \ell \cdot R^{\ell-1}\right] \cdot P_\ell(\cos\theta) = -4\pi\sigma$$

$\hookrightarrow$

$$\rightsquigarrow \sum_{\ell} (2\ell+1) A_\ell R^{\ell-1} P_\ell(\cos\theta) = 4\pi\sigma \qquad \Big| \cdot \int_0^\pi P_{\ell'}(\cos\theta)\, d\theta$$

$$\sum_{\ell} (2\ell+1) A_\ell R^{\ell-1} \int P_{\ell'}(\cos\theta) P_\ell(\cos\theta)\, d\theta = 4\pi \int_0^\pi k \cdot \underbrace{\cos(\theta)}_{=P_1} P_{\ell'}(\cos\theta)\, d\theta$$

$$(2\ell'+1) A_{\ell'} R^{\ell'-1} \frac{2}{2\ell'+1} = 4\pi k \int_0^\pi \underbrace{P_1 \cdot P_{\ell'} \cdot d\theta}_{\delta_{1\ell'}}$$

$$A_{\ell'} R^{\ell'-1} = 2\pi k \cdot \frac{2}{2\ell'+1} \cdot \delta_{1\ell'}$$

$$\rightsquigarrow A_\ell = \frac{4\pi k}{2\ell+1} \cdot \frac{1}{R^{\ell-1}} \cdot \delta_{1\ell}$$

For inside:

$$V(r,\theta) = \sum_{\ell} \frac{4\pi k}{2\ell+1} \cdot \frac{1}{R^{\ell-1}} \delta_{1\ell} \cdot r^\ell P_\ell(\cos\theta) \qquad \Big| \text{ only } \ell = 1 \text{ provide solution!}$$

$$= \frac{4\pi k}{3} \cdot \frac{r}{R^0} P_1(\cos\theta)$$

$$= \frac{4\pi}{3} k \cdot r \cdot \cos\theta$$

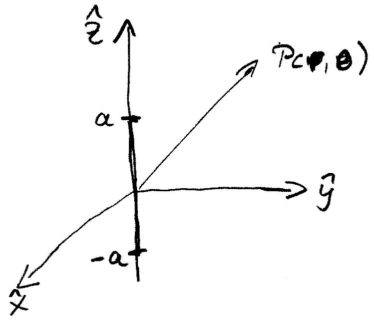
For outside:

$$B_\ell = A_\ell \cdot R^{2\ell+1} = \frac{4\pi k}{2\ell+1} \cdot \frac{R^{2\ell+1}}{R^{\ell-1}} \delta_{1\ell} = \frac{4\pi k}{2\ell+1} R^{\ell+2} \delta_{1\ell}$$

$$\hookrightarrow V(r,\theta) = \sum_{\ell} B_\ell r^{-(\ell+1)} P_\ell(\cos\theta) = \sum \frac{4\pi k}{2\ell+1} R^{\ell+2} r^{-(\ell+1)} \delta_{1\ell} P_\ell(\cos\theta)$$

$$= \frac{4\pi}{3} k \cdot \frac{R^3}{r^2} \cdot \cos\theta$$

Problem #2: *(20 points)* A line charge density is distributed on the z-axis from z = -a to z = a. Using the method of Green's function, find the potential for r > a to the order of $(a/r)^5$.



line charge

$$\rho(\vec{r}') = \lambda \cdot \frac{1}{2\pi r'^2} \cdot \left[\delta(\cos\theta' - \cos\phi) + \delta(\cos\theta' - \cos\pi) \right]$$

$$Q = \int_V \rho(r')\, d^3r' = \lambda \int_V \frac{1}{2\pi r'^2} \cdot \left[\delta(\cos\theta' - 1) + \delta(\cos\theta' - \cos\pi) \right] \cdot r'^2 dr' \sin\theta' d\theta' d\varphi$$

$$= \frac{\lambda}{2\pi} \int_0^{2\pi} d\varphi' \int_0^a dr' \int_0^\pi \left[\delta(\cos\theta' - 1) + \delta(\cos\theta' + 1) \right] \cdot \sin\theta' d\theta'$$

$$= \lambda \cdot a \left[\int_0^\pi \delta(\cos\theta' - 1) \cdot \sin\theta' d\theta' + \int_0^\pi \delta(\cos\theta' + 1) \sin\theta' d\theta' \right]$$

$$= \lambda \cdot a \cdot \left[\underbrace{\int_{-1}^{0} \delta(x-1)\, dx}_{=1} + \underbrace{\int_{-1}^{0} \delta(x+1)\, dx}_{=1} \right]$$

$$= \underline{\underline{2\lambda \cdot a}}$$

$$\hookrightarrow \quad \boxed{\lambda = \frac{Q}{2a}}$$

Potential: $\boxed{V(\vec{r}) = \int_V \rho(\vec{r}')\, G(\vec{r}, \vec{r}')\, d^3r'}$

Greens function for no boundary:

$$G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} = 4\pi \sum_{lm} \frac{r_<^l}{r_>^{l+1}} \cdot Y^*_{lm}(\mu', \varphi')\, Y_{lm}(\mu, \varphi) \Longrightarrow$$

$$Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\cdot\frac{(\ell-m)!}{(\ell+m)!}}\cdot P_{\ell m}(\cos\theta)\, e^{im\varphi}$$

$r > r' = a$ and azimthal symmetry $(m=0)$

$$\hookrightarrow Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}}\cdot P_\ell(\theta)$$

$$\hookrightarrow V(\vec r) = \int_V \lambda\cdot\frac{1}{2\pi r'^2}\left[\delta(\cos\theta'-1)+\delta(\cos\theta'+1)\right]\cdot 4\pi\cdot\sum_\ell \frac{1}{2\ell+1}\,\frac{r'^\ell}{r^{\ell+1}}\cdot P_\ell(\theta')$$

$$\cdot\frac{2\ell+1}{4\pi}\,P_\ell(\theta)\cdot r'^2\cdot dr'\sin\theta'\,d\theta' d\varphi'$$

$$= \sum_\ell \lambda\cdot\frac{1}{r^{\ell+1}}P_\ell(\theta)\cdot\int_0^a r'^\ell dr'\cdot\int_0^\pi\left[\delta(\cos\theta'-1)+\delta(\cos\theta'+1)\right]\cdot P_{\ell'}(\theta')\sin\theta' d\theta$$

$$= \sum_\ell \frac{\lambda\cdot P_\ell(\theta)}{r^{\ell+1}}\cdot\frac{r'^{\ell+1}}{\ell+1}\Big|_0^a\cdot\left[\int_0^\pi\delta(\cos\theta'-1)P_\ell(\theta')\sin\theta' d\theta' + \int_0^\pi\delta(\cos\theta'+1)P_\ell(\theta')\cdot\sin\theta' d\theta'\right]$$

$$= \sum_\ell \frac{\lambda\cdot P_\ell(\theta)}{r^{\ell+1}}\cdot\frac{a^{\ell+1}}{\ell+1}\left[P_\ell(1)+P_\ell(-1)\right]$$

for odd ℓ: $P_\ell(x) = -P_\ell(-x)$, for even ℓ: $P_\ell(x) = P_\ell(x)$

$$\rightsquigarrow V(\vec r) = \sum_{\text{even}} \frac{\lambda\cdot P_\ell(\cos\theta)}{r^{\ell+1}}\,\frac{a^{\ell+1}}{\ell+1}\cdot 2P_\ell(1)$$

$$= \frac{\lambda P_0(\cos\theta)}{r}\cdot a\cdot 2P_0(1) + \frac{\lambda\cdot P_2(\cos\theta)}{r^3}\,\frac{a^3}{3}\cdot 2P_2(1) + \frac{\lambda P_4(\cos\theta)}{r^5}\,\frac{a^5}{5}\cdot 2P_4(1)$$

$$= 2\cdot\lambda\cdot\left[\frac{a}{r}P_0(\cos\theta) + \frac{1}{3}\left(\frac{a}{r}\right)^3 P_2(\cos\theta) + \frac{1}{5}\left(\frac{a}{r}\right)^5 P_4(\cos\theta) + \cdots\right]$$

Problem 3*: (20 points)* (A sphere of radius R carries the charge density $\rho(\vec{r}) = \frac{A(1 - \cos\theta)}{r^2}$, where A is a constant. Find the potential far from the sphere.
)

Sphere with radius R and charge density $\rho(\vec{r}) = \frac{A\cdot(1-\cos\theta)}{r^2}$

Find φ far from sphere!

$$\rho(r,\theta) = \frac{A}{r^2}\left[P_0(\cos\theta) - P_1(\cos\theta)\right]$$

A problem with azimuthal symmetry $\rightarrow$ find potential on z-axis $\rightsquigarrow$ $z \rightarrow r \times P_\ell(\cos\theta)$

$$\varphi(z) = \int \frac{\rho(\vec{r}')\,d^3\vec{r}'}{4\pi\varepsilon_0\,|\vec{r}-\vec{r}'|} = \frac{1}{4\pi\varepsilon_0}\iiint \frac{A}{r^2}(P_0 - P_1)\frac{r^2\sin\theta\,d\theta\,d\varphi\,dr}{\sqrt{z^2+r^2-2z\cdot r\cos\theta}}$$

integr. over φ give the factor 2π

and

$$\frac{1}{\sqrt{z^2+r^2-2zr\cos\theta}} = \frac{1}{z}\cdot\frac{1}{\sqrt{1+\left(\frac{r}{z}\right)^2 - 2\cdot\frac{r}{z}\cos\theta}} = \frac{1}{z}\sum_{\ell=0}^{\infty}\left(\frac{r}{z}\right)^\ell P_\ell(\cos\theta)$$

$$\hookrightarrow \varphi(z) = \frac{A\cdot 2\pi}{4\pi\varepsilon_0}\int_0^R\int_0^\pi dr\,(P_0(\cos\theta) - P_1(\cos\theta))\cdot\frac{1}{z}\sum_{\ell=0}^{\infty}\left(\frac{r}{z}\right)^\ell P_\ell(\cos\theta)\cdot d(\cos\theta)$$

$$= \frac{A}{2\varepsilon_0}\sum_{\ell=0}^{\infty}\int_0^R \frac{1}{z}\cdot\left(\frac{r}{z}\right)^\ell\cdot\int_{-1}^{1}\left(P_0(x) - P_1(x)\right)P_\ell(x)\,dx\,dr$$

we know $\int_{-1}^{1} P_\ell\cdot P_{\ell'}\,dx = \frac{2}{2\ell+1}\cdot\delta_{\ell\ell'}$

$$\rightsquigarrow \int_{-1}^{1} P_0\cdot P_0\,dx = 2, \quad \int P_1 P_1\,dx = \frac{2}{3}$$

and $\int_{-1}^{1}(P_0 - P_1)P_\ell\,dx = 0$ if $\ell \geq 2$ $\Rightarrow$

$$\leadsto \varphi(z) = \frac{A}{2\varepsilon_0} \int_0^R \frac{1}{z}\left[\left(\frac{r}{z}\right)^0 \cdot 2 - \left(\frac{r}{z}\right)^1 \cdot \frac{2}{3} \right] dr$$

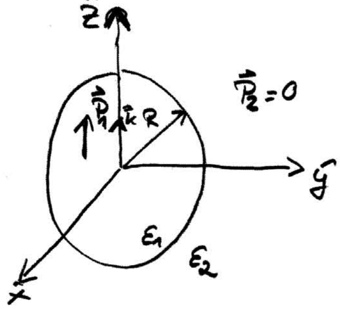
$$= \frac{A}{\varepsilon_0} \cdot \frac{1}{z} \cdot \int_0^R \left[1 - \frac{1}{3}\frac{r}{z} \right] dr$$

$$= \frac{A}{\varepsilon_0 \cdot z} \left(R - \frac{1}{6}\frac{R^2}{z} \right) = \underline{\underline{\frac{A \cdot R}{\varepsilon_0} \left(\frac{1}{z} - \frac{1}{6}\frac{R}{z^2} \right)}} \quad (z>0)$$

$$\varphi(z,\theta) = \frac{A R}{\varepsilon_0} \left[\frac{1}{r} P_0(\cos\theta) - \frac{1}{6} R \cdot \frac{1}{r^2} P_1(\cos\theta) \right]$$

$$= \frac{A \cdot R}{\varepsilon_0} \left(\frac{1}{r} - \frac{R}{6r^2} \cos\theta \right)$$

Problem 4: *(20 points): Find the electric field inside and outside a dielectric sphere of radius R that has a uniform polarization vector P.*



Dielectric sphere with ε_1 - inside
ε_2 - outside

Uniform polarization $\vec{P}$ say parallel to z-axis, with $\hat{k} = (0,0,z)$
$\rightsquigarrow \vec{P}_1 = P \cdot \hat{k}$

General solution of a problem with azimuthal symmetry:

$$\varphi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l \cdot r^l + B_l \, r^{-(l+1)}\right] P_l(\cos\theta)$$

Inside of sphere: $r \leq R$: $\varphi_{in}(r,\theta) = \sum_l A_l r^l P_l(\cos\theta)$

Outside $r \geq R$: $\varphi_{out}(r,\theta) = \sum_l B_l \, r^{-(l+1)} P_l(\cos\theta)$

at the sphere: $\varphi_{in}(R,\theta) = \varphi_{out}(R,\theta)$

$$\rightsquigarrow \sum_l A_l R^l P_l(\cos\theta) = \sum_l B_l \, R^{-(l+1)} P_l(\cos\theta)$$

$$l: \quad A_l R^l = B_l \, R^{-(l+1)} \rightsquigarrow \boxed{B_l = A_l \cdot R^{2l+1}}$$

$$\vec{D}_2 \cdot \hat{n} - \vec{D}_1 \cdot \hat{n} = \sigma = -(\vec{P}_2 - \vec{P}_1) \cdot \hat{n}, \quad \vec{P}_1 = P \cdot \underbrace{\hat{k}}_{\cos\theta}, \quad \vec{P}_2 = 0$$

$$\rightsquigarrow (\varepsilon_2 \cdot \vec{E}_2 - \varepsilon_1 \cdot \vec{E}_1) \cdot \hat{n} = \varepsilon_2 \cdot \frac{\partial \varphi_2}{\partial r}\Big|_R - \varepsilon_1 \frac{\partial \varphi_1}{\partial r}\Big|_R = -\sigma = P \cdot \cos\theta$$

$$\rightsquigarrow \varepsilon_2 \cdot \sum_l B_l \cdot -(l+1) R^{-(l+2)} \cdot P_l - \varepsilon_1 \sum_l A_l \cdot l \cdot R^{l-1} P_l = P \cdot \cos\theta$$

$A_l = B_l \cdot R^{-(2l+1)}$ $\qquad = P \cdot P_1(\cos\theta)$

$$\varepsilon_2 \sum_l B_l \cdot (-l-1) R^{-(l+2)} P_l - \varepsilon_1 \cdot \sum_l B_l \cdot l \cdot R^{-(l+2)} \cdot P_l = -P \cdot P_1(\cos\theta)$$

with $\int_{-1}^{1} P_l P_{l'} = \frac{2}{2l+1} \delta_{ll'} \rightsquigarrow$

$l = 0$: $B_0 = 0$, $A_0 = 0$ $\qquad \Longrightarrow$

$$l=1: \quad -2\varepsilon_2 B_1 R^{-3} P_1 - \varepsilon_1 \cdot B_1 R^{-3} \cdot P_1 = -p \cdot P_1$$

$$\rightsquigarrow B_1 = \frac{p \cdot R^3}{(2\varepsilon_2 + \varepsilon_1)}, \quad A_1 = \frac{p}{2\varepsilon_2 + \varepsilon_1}$$

$$l > 1: \quad B_l = A_l = 0$$

$$\hookrightarrow \varphi_{out} = \frac{p \cdot R^3}{2\varepsilon_2 + \varepsilon_1} \cdot \frac{1}{r^2} \cdot P_1(\cos\theta) = \frac{1}{2\varepsilon_2 + \varepsilon_1} \cdot p \cdot R^3 \frac{\cos\theta}{r^2}$$

$$\varphi_{in} = \frac{p}{2\varepsilon_2 + \varepsilon_1} \cdot r \cdot P_1(\cos\theta) = \frac{1}{2\varepsilon_2 + \varepsilon_1} p \cdot r \cdot \cos\theta$$

$$E_{r\,out} = -\frac{\partial \varphi_{out}}{\partial r} = -\frac{2}{2\varepsilon_2 + \varepsilon_1} p \cdot R^3 \frac{\cos\theta}{r^3}$$

$$E_{\theta\,out} = -\frac{1}{r}\frac{\partial \varphi_{out}}{\partial \theta} = \frac{1}{2\varepsilon_2 + \varepsilon_1} \cdot \frac{p R^3}{r^3} \sin\theta$$

$$E_{r\,in} = -\frac{\partial \varphi_{in}}{\partial r} = -\frac{p \cdot \cos\theta}{2\varepsilon_2 + \varepsilon_1}$$

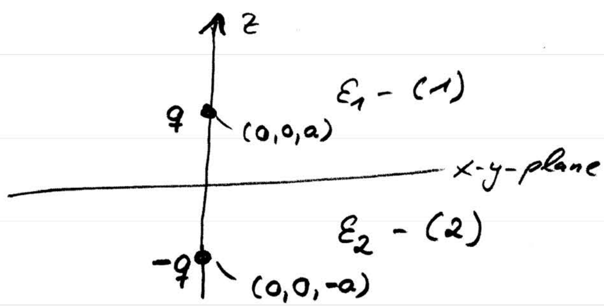
$$E_{\theta\,in} = -\frac{1}{r}\frac{\partial \varphi_{in}}{\partial \theta} = -\frac{1}{2\varepsilon_2 + \varepsilon_1} \cdot p \cdot A \sin\theta$$

$$E_{z\,in} = -\frac{\partial \varphi_{in}}{\partial z} = \frac{p}{2\varepsilon_2 + \varepsilon_1} = \text{constant} - \text{uniform electro field}$$

$$E_x = E_x = 0 \quad (\text{inside})$$

outside: $\varphi = \frac{1}{2\varepsilon_2 + \varepsilon_1} p \cdot r^3 \cdot \frac{\vec{r} \cdot \hat{z}}{r^3}$ — field of dipole

Problem 5: *(20 points): A plane interface separates two semi-infinite dielectric media with dielectric constants ε_1 and ε_2. The surface may be taken as the plane z = 0. A charge q in the dielectric ε_1 is at (0, 0, a) and a charge –q in the dielectric ε_2 is at (o, 0, -a). Find the forces between these two charges (hint: use the method of images and apply the proper boundary conditions).*



Force on charge q in region (1) is due to induced charges (due to q) and charge (−q) in region (2).

Induced charges $\Leftrightarrow$ image charge of q : $Q_1 = q \cdot \frac{(\varepsilon_1 - \varepsilon_2)}{\varepsilon_1 + \varepsilon_2}$ @ (0,0,−a)

Electric field in (1) due to charge (−q) → image charge

$$Q_2 = (-q) \cdot \frac{2\varepsilon_1}{\varepsilon_2 + \varepsilon_1}$$

Electric field @ (0,0,a) acting on q :

$$E_1 = \frac{Q_1 + Q_2}{4\pi\varepsilon_1 (2a)^2} \rightsquigarrow \text{Force } F_1 = q \cdot E = \frac{q \cdot (Q_1 + Q_2)}{4\pi\varepsilon_1 (2a)^2}$$

$$F_1 = \frac{q^2}{16\pi\varepsilon_1 a^2} \cdot \left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} - \frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2} \right) = -\frac{q^2}{16\pi\varepsilon_1 a^2}$$

Similarly : $F_2 = \frac{-q^2}{16\pi\varepsilon_2 a^2}$ - force on charge (−q) local charge density $\varrho(x,y,z)$, placed in an extern. electr.-static field - described by $\Phi^{(0)}(x,y,z)$.

a) The force on the charge distribution with $\varrho(\vec{r})$ placed in an external field $E^{(0)}(\vec{r})$ is

$$\vec{F} = \int \varrho(\vec{r})\, E^{(0)}(\vec{r})\, d^3\vec{r}$$

Since the field varies slowly we can expand each component of the electr. field:

$$E_i^{(0)}(\vec{r}) = E_i^{(0)}(0) + \sum_j x_j \left(\frac{\partial E_i^{(0)}(\vec{y})}{\partial y_j}\right)_{\vec{y}=0} + \frac{1}{2}\sum_{j,k} x_j \cdot x_k \cdot \left(\frac{\partial^2 E_i^{(0)}(\vec{y})}{\partial y_j\, \partial y_k}\right)_{\vec{y}=0} + \dots$$

Now multiply by $\varrho(\vec{r})$ and integrate

$$F_i = E_i^{(0)}(0) \cdot \underbrace{\int \varrho(\vec{r})\, d^3\vec{r}}_{=q} + \sum_j \left(\frac{\partial E^{(0)}(\vec{y})}{\partial y_j}\right)_{\vec{y}=0} \cdot \underbrace{\int x_j\, \varrho(\vec{r})\, d^3\vec{r}}_{=P_j}$$

$$+ \frac{1}{2}\sum_{j,k}\left(\frac{\partial^2 E_i^{(0)}(\vec{y})}{\partial y_j\, \partial y_k}\right)_{\vec{y}=0} \cdot \int \underbrace{x_j \cdot x_k\, \varrho(\vec{r})\, d^3\vec{r}}_{=\frac{1}{3} Q_{jk}} + \dots$$

$$\leadsto \quad \vec{F} = q \cdot E_i^{(0)}(0) + \underbrace{\vec{p} \cdot \nabla_y E_i^{(0)}(\vec{y})\Big|_{\vec{y}=0}}_{=\nabla(\vec{p}\cdot E_i^{(0)})|_{x=0}} + \nabla\left[\frac{1}{6}\sum_{jk} Q_{jk} \cdot \frac{\partial E_j}{\partial x_k}\right] + \dots$$

$$= q \cdot E_i^{(0)} + \nabla(\vec{p}\cdot\vec{E}^{(0)}) + \nabla\left[\frac{1}{6}\sum_{jk} Q_{jk}\, \frac{\partial E_j}{\partial x_k}\right]$$

$\Rightarrow$