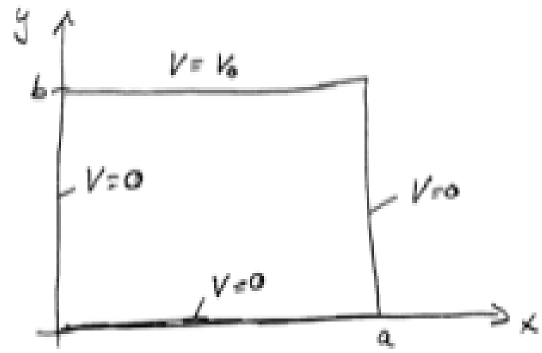
Solutions for Homework # 7

Problem #1: *(30 points)* An infinitely long rectangular conductive cylinder has three sides grounded, while the fourth (imagined to be insulated from others by very tine gapes) is held at potential V_o. Find the potential at all interior points.!



$x = 0, \quad V = 0$
$x = a, \quad V = 0$
$y = 0, \quad V = 0$
$y = b \quad V = V_0$

Laplace Equ. $\boxed{\Delta V = 0}$

two dim. problem $\quad V = V_x(x) \cdot V_y(y)$

$$\leadsto \Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V_x(x)}{\partial x^2} \cdot V_y + \frac{\partial^2 V_y}{\partial y^2} \cdot V_x(x) = 0$$

$$\leadsto \Delta V_x + \alpha^2 V_x = 0 \quad \text{And} \quad \Delta V_y - \alpha^2 V_y = 0$$

$$\hookrightarrow V_x = A \cdot \sin(\alpha x) + B \cos(\alpha \cdot x)$$

$$V_y = C \cdot e^{\alpha \cdot y} + D \cdot e^{-\alpha y}$$

Boundary condition

$$V_x(x=0) = 0 = A \cdot \sin(0) + B \cdot \cos(0) \Rightarrow B = 0$$

$$V_x(x=a) = 0 = A \cdot \sin(\alpha a) \leadsto \alpha \cdot a = n \cdot \pi \leadsto \alpha = \frac{n\pi}{a}$$

$$\hookrightarrow V_x(x) = A_n \cdot \sin\left(\frac{n \cdot \pi}{a} \cdot x\right)$$

$$V_y(y=0) = 0 = C \cdot e^0 + D \cdot e^{-0} \leadsto C = -D$$

$$V_y(y=b) = V_0 = C \cdot \left[e^{+\alpha \cdot b} - e^{-\alpha \cdot b}\right] = C \cdot \sinh(\alpha \cdot b)$$

$$\hookrightarrow \text{general solution} \quad V = V_x \cdot V_y = \sum_{n=1}^{\infty} \left[A_n \cdot \sin\left(\frac{n\pi}{a} \cdot x\right)\right] \cdot \left[C_n \cdot \sinh\left(\frac{n\pi}{a} \cdot y\right)\right]$$

$\Rightarrow$

back to boundary conditions:

$$V(x, y=b) = V_0 = \sum_{n=1}^{\infty} \underbrace{A_n \cdot C_n}_{E_n} \cdot \sin\left(\frac{n\pi}{a}x\right) \cdot \sinh\left(\frac{n\pi}{a} \cdot b\right) \Big/ \cdot \int_0^a \sin\left(\frac{n'\pi}{a}x\right)dx$$

$$\rightsquigarrow V_0 \int_0^a \sin\left(\frac{n'\pi}{a} \cdot x\right)dx = \sum_{n=1}^{\infty} E_n \cdot \sinh\left(\frac{n\pi}{a} \cdot b\right) \cdot \underbrace{\int_0^a \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n'\pi}{a}x\right)dx}_{=\frac{a}{2}\delta(n-n')}$$

$$\hookrightarrow \underbrace{V_0 \int_0^a \sin\left(\frac{n'\pi}{a}x\right)dx}_{= \left.\frac{\cos\left(\frac{n'\pi x}{a}\right)}{\left(\frac{n'\pi}{a}\right)}\right|_0^a} = E_n' \sinh\left(\frac{n'\pi}{a} \cdot b\right) \cdot \frac{a}{2}$$

$$= E_n' \cdot \frac{a}{2} \sinh\left(\frac{n'\pi b}{a}\right)$$

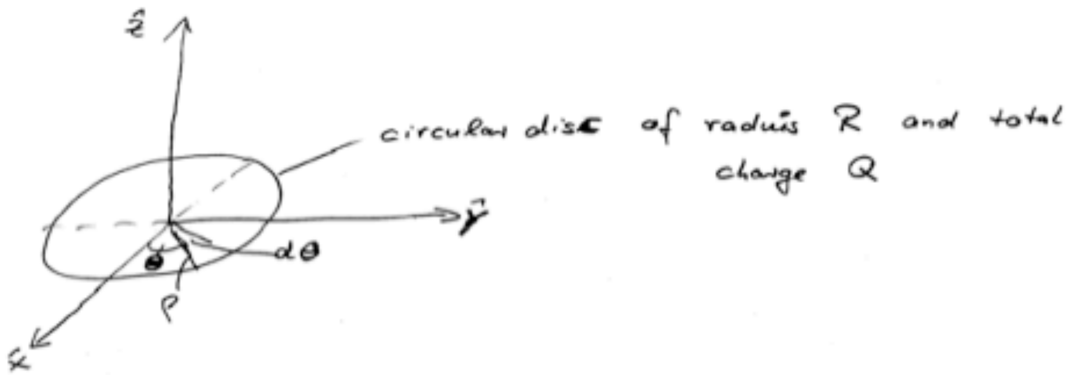
For $n' =$ even integers: $E_n' = 0$

$n' =$ odd " : $E_n' = \frac{4V_0}{n'\pi}\left[\sinh\left(\frac{n'\pi b}{a}\right)\right]$

$\hookrightarrow$ compl. solution

$$\boxed{\phi(x,y) = \sum_{n\,\text{odd}} \frac{4V_0}{n \cdot \pi} \frac{1}{\sinh\left(\frac{\pi n b}{a}\right)} \cdot \sin\left(\frac{n\cdot\pi}{a} x\right) \cdot \sinh\left(\frac{n\pi}{a} \cdot y\right)}$$

Problem #2: *(20 points):* Find the potential and the electric field strength along the axis of a thin uniformly charged circular disc of radius *R* and total charge *q*. Show that the normal component of the field changes by σ/ε_0 on passing through the surface of the disc. Consider the field at large distances from the disc.



potential $V(r) = \dfrac{1}{4\pi\varepsilon_0} \dfrac{\sigma(\rho \cdot d\rho \cdot d\theta)}{|\vec{r}|}$

$$V(z) = \int_0^{2\pi}\int_0^{R} V(r) = \int_0^{2\pi}\int_0^{R} \frac{1}{4\pi\varepsilon_0} \frac{\sigma\, \rho \cdot d\rho\, d\theta}{\sqrt{\rho^2 + z^2}}$$

$$= \frac{\sigma}{4\pi\varepsilon_0}\, 2\pi \sqrt{\rho^2+z^2}\,\Big|_0^R = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+z^2} - z\right]$$

$$E = -\nabla V \Rightarrow E_z = \frac{\partial}{\partial z} V \cdot \hat{z} = -\frac{\sigma}{2\varepsilon_0}\left[\frac{1}{2}(R^2+z^2)^{-1/2} \cdot 2 \cdot z - 1\right]\hat{z}$$

$$= \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right] \cdot \hat{z}$$

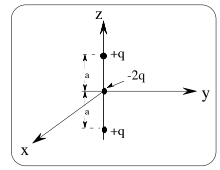
$$E_{0+} = \frac{\sigma}{2\varepsilon_0} \cdot \hat{z}\ , \qquad E_{0-} = -\frac{\sigma}{2\varepsilon_0}\hat{z}$$

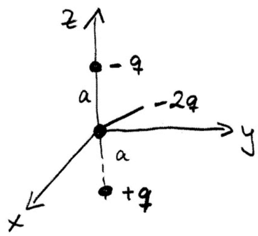
$$\hookrightarrow\ |E_{0+} - E_{0-}| = \frac{\sigma}{2\varepsilon_0} - \left(-\frac{\sigma}{2\varepsilon_0}\right) = \underline{\underline{\frac{\sigma}{\varepsilon_0}}}$$

Problem #3: *(20 points)*

(a) Find the first two sets of non-vanishing moments, q_{lm}, for the following charge distribution

(b) Write down the multipole expansion of the potential due to this charge distribution, keeping only the first two sets of q_{lm}.





$$\rho(\vec{r}) = \frac{q}{2\pi r^2}\Big[-\delta(r) + \delta(r-a)\cdot\delta(\cos\theta - 1) + \delta(r-a)\,\delta(\cos\theta + 1)\Big]$$

$$q_{\ell m} = \int_V d^3r'\, \rho(r')\, r'^{\ell}\, Y^*_{\ell m}(\theta', \varphi')$$

a.) We have azimuthal symmetry : $m = 0$

$$Y_{\ell m}(\theta, 0) = \sqrt{\frac{2\ell+1}{4\pi}}\cdot P_\ell(\cos\theta)$$

$$\hookrightarrow\; q_{\ell 0} = \int d^3r'\, \rho(r')\, r'^{\ell}\sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta')$$

for $\ell = 0$:
$$q_{00} = \frac{1}{\sqrt{4\pi}}\cdot\frac{q}{2\pi}\int\frac{1}{r'^2}\Big[-\delta(r') + \delta(r'-a)\,\delta(\cos\theta'-1) + \delta(r'-a)\,\delta(\cos\theta'+1)\Big]\cdot P_0(\cos\theta')\cdot r'^2\cdot dr'\cdot\sin\theta'\, d\theta'\, d\varphi'$$

$$= \frac{q}{\sqrt{4\pi}}\Big[-\int_0^a\delta(r')\,dr'\int_0^\pi\sin\theta\, d\theta + \int_0^a\delta(r'-a)\,dr'\int_0^\pi\delta(\cos\theta'-1)\sin\theta'\,d\theta' + \int_0^a\delta(r'-a)\,dr'\int_0^\pi\delta(\cos\theta'+1)\sin\theta'\,d\theta'\Big]$$

$= -2 + 1 + 1 = 0$: monopole (net charge = zero)

non-vanishing moments may be found from the general term for $q_{\ell 0}$, $\ell \neq 0$

$$q_{\ell 0} = \frac{q}{2\pi}\sqrt{\frac{2\ell+1}{4\pi}}\int_0^{2\pi}d\varphi\int_0^{a}\int_0^{\pi}\Big[-\delta(r') + \delta(r'-a)\,\delta(\cos\theta'-1) + \delta(r'-a)\,\delta(\cos\theta'+1)\Big]\times P_\ell(\cos\theta')\, r'^{\ell}\, dr'\sin\theta'\, d\theta' \Longrightarrow$$

$$\rightsquigarrow q_{\ell 0} = \sqrt{\frac{2\ell+1}{4\pi}} \cdot q \cdot \left[(0)^\ell + a^\ell P_\ell(1) + a^\ell P_\ell(-1) \right]$$

$$= q \sqrt{\frac{2\ell+1}{4\pi}}\, a^\ell \Big[\underbrace{P_\ell(1) + P_\ell(-1)}_{\substack{=0 \text{ for odd } \ell \\ =2 \text{ " even } \ell}} \Big]$$

$$= 2q \sqrt{\frac{2\ell+1}{4\pi}}\, a^\ell \quad \text{for even } \ell \quad (\text{since } P_\ell(1) = 1 \text{ for all } \ell)$$

$$\ell = 2: \quad q_{20} = 2q\sqrt{\frac{5}{4\pi}}\, a^2 = q\, a^2 \cdot \sqrt{\frac{5}{\pi}} \qquad (*)$$

$$\ell = 4: \quad q_{40} = 2q\sqrt{\frac{9}{4\pi}}\, a^4 = q \cdot a^4 \cdot \frac{3}{\sqrt{\pi}}$$

b.)

$$V(\vec{r}) = 4\pi \sum_\ell \frac{1}{2\ell+1}\, q_{\ell 0}\, \frac{1}{r^{\ell+1}} \sqrt{\frac{2\ell+1}{4\pi}} \cdot P_\ell(\cos\theta) \qquad \Big|\ m=0!$$

$$\ell = 2,4: \quad \rightsquigarrow V(\vec{r}) = 4\pi \left[\frac{1}{5}\, q_{20}\, \frac{1}{r^3} \sqrt{\frac{5}{4\pi}}\, P_2(\cos\theta) + \frac{1}{9}\, q_{40}\, \frac{1}{r^5} \sqrt{\frac{9}{4\pi}}\, P_4(\cos\theta) + \dots \right]$$

$$= 4\pi \cdot \left[\frac{1}{5}\, q\, a^2 \sqrt{\frac{5}{\pi}} \cdot \sqrt{\frac{5}{4\pi}}\, P_2(\cos\theta) + \frac{1}{9}\, q \cdot a^4 \frac{3}{\sqrt{\pi}}\, \frac{1}{r^5} \sqrt{\frac{9}{4\pi}}\, P_4(\cos\theta) + \dots \right]$$

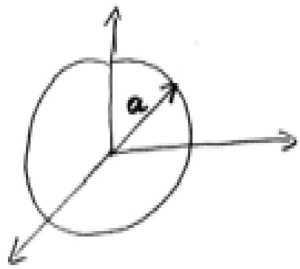
$$\rightsquigarrow \quad V(\vec{r}) = \frac{2q}{a} \cdot \left[\left(\frac{a}{r}\right)^3 P_2(\cos\theta) + \left(\frac{a}{r}\right)^5 P_4(\cos\theta) + \dots \right]$$

(*) Momentums:

$$P = \int \vec{r}\,' \rho(\vec{r}\,')\, d^3r = \dots = 0$$

$$Q_{ij} = \int \rho(\vec{r}\,') \left(3 x_i' x_j' - \delta_{ij} r'^2 \right) d^3 r' = \dots \begin{pmatrix} -qa^2 & & 0 \\ & -qa^2 & \\ 0 & & 2qa^2 \end{pmatrix}$$

Problem #4: *(20 points)* - Jackson problem 3.5. A hollow sphere of an inner radius ‘a’ has the potential specified on its surface to be . Prove the equivalence of the two forms of the solution for the potential inside the sphere …..



a) $G(\vec{r},\vec{r}') = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{1}{\sqrt{\frac{r^2r'^2}{a^2}+a^2-2rr'\cos\gamma}}$

$$\frac{\partial G}{\partial n'} = -\frac{r^2-a^2}{a\,(r^2+a^2-2ar\cos\gamma)^{3/2}}$$

Potential $V(\vec{r}) = -\frac{1}{4\pi}\oint_S V(\vec{r}')\,\frac{\partial G}{\partial n'}\,d^2r' = \frac{1}{4\pi}\int \frac{V(\theta',\varphi')\,a\,(a^2-r^2)}{(r^2+a^2-2ar\cos\gamma)^{3/2}}\,d\Omega'$

$$= \frac{a\,(a^2-r^2)}{4\pi}\int \frac{V(\theta',\varphi')}{(r^2+a^2-2ar\cos\gamma)^{3/2}}\,d\Omega$$

b) we know that

$$V(r,\theta,\varphi) = \sum_{l,m}\left[A_{lm}\,r^l + B_{lm}\cdot r^{-(l+1)}\right]\cdot Y_{lm}(\theta,\varphi)$$

at $r=a$: $V = V_0(\theta,\varphi)$

Potential inside $V(r,\theta,\varphi) = \sum_{lm} A_{lm}\,r^l\,Y_{lm}(\theta,\varphi)$

at $r=a$ $V = V_0(\theta,\varphi) = \sum_{l,m} A_{lm}\,a^l\,Y_{lm}(\theta,\varphi)$

$$\rightsquigarrow A_{lm} = \int \frac{V_0(\theta',\varphi')}{a^l}\,Y^*_{lm}(\theta',\varphi')\,d\Omega'$$

$$\Rightarrow V(r,\theta,\varphi) = \sum_{l,m}\left[\int \frac{V_0(\theta',\varphi')}{a^l}\,Y^*_{lm}(\theta',\varphi')\cdot r^l\,Y_{lm}(\theta,\varphi)\right]d\Omega'$$

$$= \sum_{l,m} V_0(\theta',\varphi')\left(\frac{r}{a}\right)^l\cdot Y^*_{lm}(\theta',\varphi')\cdot Y_{lm}(\theta,\varphi)\,d\Omega' \leftarrow (1)$$

for $r \to z$: $\gamma = 0$; $\theta \to 0$

$$\frac{a\cdot(a^2-z^2)}{(z^2+a^2-2a\cdot z\cos\gamma)^{3/2}} = \frac{a^3(1-z^2/a^2)}{a^3\left[\left(1-\frac{z}{a}\right)^2\right]^{3/2}} = \frac{1+\frac{z}{a}}{\left(1-\frac{z}{a}\right)^2} = \left(1+\frac{z}{a}\right)\left(1-\frac{z}{a}\right)^{-2}$$

$$= \left(1+\frac{z}{a}\right)\cdot\left[1+2\frac{z}{a}+3\left(\frac{z}{a}\right)^2+\cdots\right]$$

$$= 1+3\cdot\frac{z}{a}+5\cdot\left(\frac{z}{a}\right)^2+\cdots$$

$$= \sum_\ell (2\ell+1)\left(\frac{z}{a}\right)^\ell$$

Now let $z \to r$; multiply by $P_\ell(\cos\theta)$ and let $\theta \to \gamma$

$$\frac{a(a^2-r^2)}{(r^2+a^2-2ar\cos\gamma)^{3/2}} = \sum_\ell (2\ell+1)\left(\frac{r}{a}\right)^\ell P_\ell(\cos\gamma)$$

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1}\sum_m Y^*_{\ell m}(\theta',\varphi')\,Y_{\ell m}(\theta,\varphi)$$

$$= \sum_\ell\sum_m \frac{4\pi}{(2\ell+1)}\cdot(2\ell+1)\cdot\left(\frac{r}{a}\right)^\ell\cdot Y^*_{\ell m}\,Y_{\ell m}$$

$$= \sum_{\ell m} 4\pi\left(\frac{r}{a}\right)^\ell Y^*_{\ell m}(\theta',\varphi')\,Y_{\ell m}(\theta,\varphi)$$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi}\int V_0(\theta',\varphi')\sum_{\ell,m} 4\pi\left(\frac{r}{a}\right)^\ell\cdot Y^*_{\ell m}\,Y_{\ell m}\,d\Omega'$$

$$= \sum_{\ell,m}\int V_0(\theta',\varphi')\left(\frac{r}{a}\right)^\ell Y^*_{\ell m}\,Y_{\ell m}\,d\Omega' = \sum_{\ell m}\left(\frac{r}{a}\right)^\ell Y_{\ell m}\underbrace{\int V_0(\theta',\varphi')Y^*_{\ell m}\,d\Omega}_{A_{\ell m}}$$

$= A_{\ell m}$

$$= \sum_{\ell,m} A_{\ell m}\left(\frac{r}{a}\right)^\ell\cdot Y_{\ell m}(\theta,\varphi) \quad = \text{equal to } (1)$$