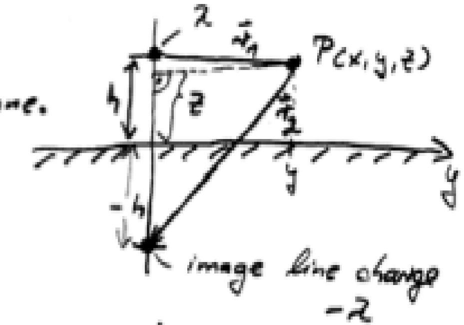
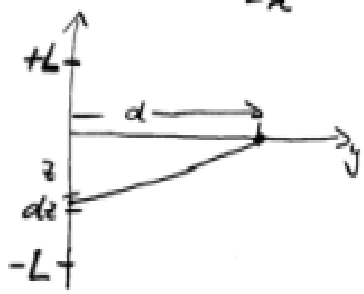
Solutions for Homework # 6

(30 points)

Problem #1:



Lets the line charge be in x-z-plane. The potential at (x,y,z) point due to the line charge λ and its image (−λ) at z = −h:



1) $$\phi(d) = \int d\phi = \int_{-\infty}^{\infty} \frac{1}{4\pi\varepsilon_0} \frac{\lambda \cdot dz}{\sqrt{z^2+d^2}}$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \ln\left(z + \sqrt{z^2+d^2}\right)\Big|_{-L}^{+L} \quad (L \to \infty)$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \ln\left[\frac{L + \sqrt{L^2+d^2}}{-L + \sqrt{L^2+d^2}}\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \ln\left[\frac{1 + \sqrt{1+(d/L)^2}}{-1 + \sqrt{1+(d/L)^2}}\right] \approx \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{2}{\frac{1}{2}\left(\frac{d}{L}\right)^2}\right) = \frac{\lambda}{4\pi\varepsilon_0} 2 \cdot \ln\left(\frac{2L}{d}\right)$$

2.) Potential ϕ (x,y,z)

$$\phi = \phi_1 + \phi_2 = \frac{\lambda}{4\pi\varepsilon_0} \cdot 2 \ln\left(\frac{2L}{r_1}\right) - \frac{\lambda}{4\pi\varepsilon_0} 2 \cdot \ln\left(\frac{2L}{r_2}\right) = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{r_2}{r_1}\right)$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \cdot \ln\left[\frac{\sqrt{y^2+(z+h)^2}}{\sqrt{y^2+(z-h)^2}}\right] = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{y^2+(z+h)^2}{y^2+(z-h)^2}\right)$$

3.) The electric field

$$E_x = -\frac{\partial \phi}{\partial x} = 0$$

$$E_y = -\frac{\partial \phi}{\partial y} = -\frac{\lambda}{4\pi\varepsilon_0} \cdot \frac{(y^2+(z-h)^2)}{(y^2+(z+h)^2)} \cdot \frac{2y\left[y^2+(z-h)^2\right] - 2y\left[y^2+(z+h)^2\right]}{\left[y^2+(z-h)^2\right]^2}$$

⟹

Problem #1 contin.

$$\rightsquigarrow E_y = -\frac{\lambda}{4\pi\varepsilon_0} \cdot \frac{2y \cdot [(z-h)^2 - (z+h)^2]}{(y^2+(z+h)^2)\cdot(y^2+(z-h)^2)}$$

$$= \frac{2\lambda}{\pi\varepsilon_0} \frac{y \cdot z \cdot h}{[y^2+(z+h)^2]\cdot[y^2+(z-h)^2]}$$

$$E_z = -\frac{\partial\phi}{\partial z} = -\frac{\lambda}{4\pi\varepsilon_0} \cdot \frac{(y^2+(z-h)^2)}{(y^2+(z+h)^2)} \cdot \frac{2(z+h)(y^2+(z-h)^2) - 2(z-h)(y^2+(z+h)^2)}{[y^2+(z-h)^2]^2}$$

$$= \frac{-\lambda}{4\pi\varepsilon_0} \cdot 2 \cdot \frac{z\cdot((z-h)^2-(z+h)^2) + h\cdot[y^2+(z-h)^2+y^2+(z+h)^2]}{[y^2+(z+h)^2]\cdot[y^2+(z-h)^2]}$$

$$= \frac{+\lambda}{2\pi\varepsilon_0} \cdot \frac{z\cdot 4 \cdot z \cdot h - 2\cdot h\cdot(y^2+z^2+h^2)}{[y^2+(z+h)^2]\cdot[y^2+(z-h)^2]}$$

4.) surface charge induced:

$$\sigma(y) = \varepsilon_0 E_z(y)\Big|_{z=0} = \frac{\varepsilon_0 \cdot \lambda}{2\pi\varepsilon_0} \cdot \frac{-2\cdot h\,(y^2+h^2)}{(y^2+h^2)^2} = \underline{\underline{\frac{-\lambda\cdot h}{\pi\cdot(y^2+h^2)}}}$$

5.) induce charge per unit length on conduct. plane

$$\int_{-\infty}^{\infty} \sigma(y)\,dy = \int_{-\infty}^{\infty} \frac{-\lambda\cdot h\cdot dy}{\pi\,(y^2+h^2)} = -\frac{\lambda\cdot h}{\pi}\cdot\frac{1}{h}\tan^{-1}\frac{y}{h}\Big|_{-\infty}^{\infty}$$

$$= -\frac{\lambda}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \underline{\underline{-\lambda}}$$

2) Jackson Problem 2.5 *(20 points)*

a) $$W = \int_r^\infty |F| dy = \frac{q^2 a}{4\pi\varepsilon_0}\int_r^\infty \frac{dy}{y^3\left(1-\frac{a^2}{y^2}\right)^2} = \frac{q^2 a}{8\pi\varepsilon_0 (r^2 - a^2)}$$

Let us compare this to disassemble the charges

$$-W' = -\frac{1}{8\pi\varepsilon_0}\sum_{i\neq j}\frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} = \frac{1}{4\pi\varepsilon_0}\left[\frac{aq^2}{r}\frac{1}{r\left(1-\frac{a^2}{r^2}\right)}\right]$$

$$= \frac{q^2 a}{4\pi\varepsilon_0 (r^2 - a^2)} > W$$

The reason for this difference is that in the first expression W, the image charge is moving and changing size, whereas in the second, whereas in the second, they don't.

b) In this case

$$W = \int_r^\infty |F| dy = \frac{q}{4\pi\varepsilon_0}\left[\int_r^\infty \frac{Q dy}{y^2} - qa^3\int_r^\infty \frac{(2y^2 - a^2)}{y(y^2 - a^2)^2} dy\right]$$

Using standard integrals, this gives

$$W = \frac{1}{4\pi\varepsilon_0}\left[\frac{q^2 a}{2(r^2 - a^2)} - \frac{q^2 a}{2r^2} - \frac{qQ}{r}\right]$$

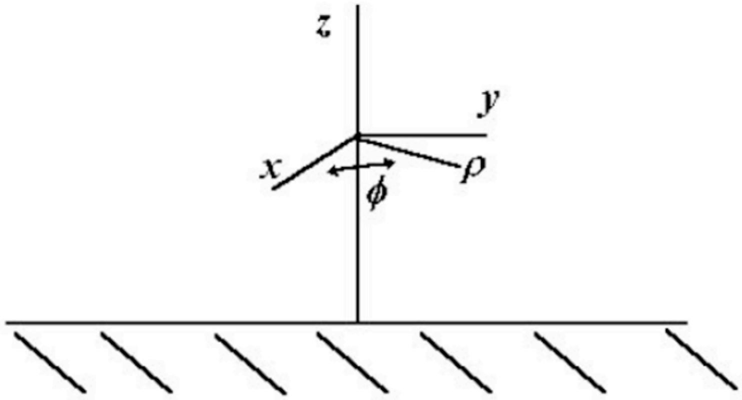
On the other hand

$$-W' = \frac{1}{4\pi\varepsilon_0}\left[\frac{aq^2}{(r^2 - a^2)} - \frac{q\left(Q + \frac{a}{r}q\right)}{r}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{aq^2}{(r^2 - a^2)} - \frac{q^2 a}{r^2} - \frac{qQ}{r}\right]$$

The first two terms are larger than those found in W for the same reason as found in a), whereas the last term is the same, because Q is fixed on the sphere.

3) Jackson Problem 2.7 *(30 points)* (see textbook chapter 2, p.87):



a) The Green's function, which vanishes on the surface is obviously

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{1}{|\vec{x}-\vec{x}_1'|}$$

where

$$\vec{x}' = x'\hat{\imath} + y'\hat{\jmath} + z'\hat{k},\ \vec{x}_1' = x'\hat{\imath} + y'\hat{\jmath} - z'\hat{k}$$

b) There is no free charge distribution, so the potential everywhere is determined by the potential on the surface. From Eq. (1.44)

$$\phi(\vec{x}) = -\frac{1}{4\pi}\int_S \phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}da'$$

Note that $\hat{n}'$ is in the $-z$ direction, so

$$\frac{\partial}{\partial n'}G(\vec{x},\vec{x}')|_{z'=0} = -\frac{\partial}{\partial z}G(\vec{x},\vec{x}')|_{z'=0} = -\frac{2z}{\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}}$$

So

$$\phi(\vec{x}) = \frac{z}{2\pi}V\int_0^a\int_0^{2\pi}\frac{\rho' d\rho' d\phi'}{\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}}$$

where $x'=\rho'\cos\phi', y' = \rho'\sin\phi'$.

c) If $\rho = 0$, or equivalently $x = y = 0$,

$$\phi(z) = \frac{z}{2\pi}V\int_0^a\int_0^{2\pi}\frac{\rho' d\rho' d\phi'}{[\rho'^2+z^2]^{3/2}} = zV\int_0^a\frac{\rho d\rho}{[\rho^2+z^2]^{3/2}}$$

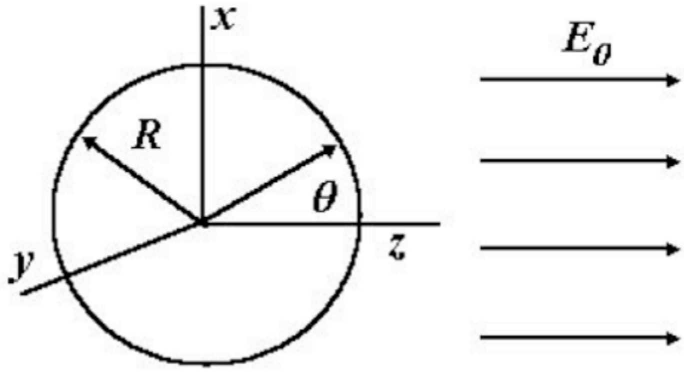
$$\phi(z) = zV\left(-\frac{z-\sqrt{(a^2+z^2)}}{z\sqrt{(a^2+z^2)}}\right) = V\left(1-\frac{z}{\sqrt{(a^2+z^2)}}\right)$$

d)

$$\phi(\vec{x}) = \frac{z}{2\pi}V\int_0^a\int_0^{2\pi}\frac{\rho' d\rho' d\phi'}{\left[(\vec{\rho}-\vec{\rho}')^2+z^2\right]^{3/2}}$$

4) Jackson Problem 2.9 *(20 points)*

The system is pictured below



a) charge density induced $\sigma = 3\varepsilon_0 E_0 \cos\theta$

the radial force/unit area outward from the surface is $\sigma^2/2\varepsilon_0$. Thus the force on the right hand hemisphere is, using $x = \cos\theta$

$$F_z = \frac{1}{2\varepsilon_0}\int \sigma^2 \hat{z}\cdot d\vec{a} = \frac{1}{2\varepsilon_0}(3\varepsilon_0 E_0)^2 2\pi R^2 \int_0^1 x^3 dx$$

$$= \frac{1}{2\varepsilon_0}(3\varepsilon_0 E_0)^2 2\pi R^2/4 = \frac{9}{4}\pi\varepsilon_0 E_0^2 R^2$$

An equal force acting in the opposite direction would be required to keep the hemispheres from sparating.

b) Now the charge density is

$$\sigma = 3\varepsilon_0 E_0 \cos\theta + \frac{Q}{4\pi R^2} = 3\varepsilon_0 E_0 x + \frac{Q}{4\pi R^2} = 3\varepsilon_0 E_0\left(x + \frac{Q}{12\pi\varepsilon_0 E_0 R^2}\right)$$

$$F_z = \frac{1}{2\varepsilon_0}\int \sigma^2 \hat{z}\cdot d\vec{a} = \frac{1}{2\varepsilon_0}(3\varepsilon_0 E_0)^2 2\pi R^2 \int_0^1 x\left(x + \frac{Q}{12\pi\varepsilon_0 E_0 R^2}\right)^2 dx$$

Thus

$$F_z = \frac{9}{4}\pi\varepsilon_0 E_0^2 R^2 + \frac{1}{2}QE_0 + \frac{1}{32\varepsilon_0\pi R^2}Q^2$$

An equal force acting in the opposite direction would be required to keep the himispheres from separating.