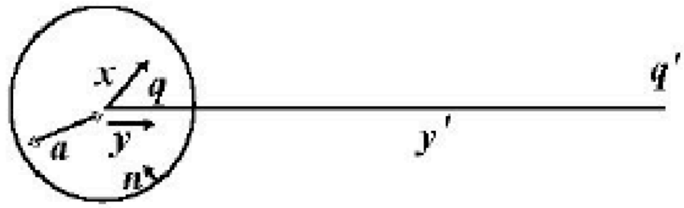
Solutions for HW # 5

(30 points)

Problem#1: (Jackson 2.2)

The system is described by



a) Using the method of images

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\vec{x}-\vec{y}|} + \frac{q'}{|\vec{x}-\vec{y}'|}\right]$$

with $y' = \frac{a^2}{y}$, and $q' = -q\frac{a}{y}$

b) $\sigma = -\varepsilon_0 \frac{\partial}{\partial n}\phi|_{x=a} = +\varepsilon_0 \frac{\partial}{\partial x}\phi|_{x=a}$

$$\sigma = \varepsilon_0 \frac{1}{4\pi\varepsilon_0}\frac{\partial}{\partial x}\left[\frac{q}{(x^2+y^2-2xy\cos\gamma)^{1/2}} + \frac{q'}{(x^2+y'^2-2xy'\cos\gamma)^{1/2}}\right]$$

$$\sigma = -q\frac{1}{4\pi}\frac{a\left(1-\frac{y^2}{a^2}\right)}{(y^2+a^2-2ay\cos\gamma)^{3/2}}$$

Note

$$q_{induced} = a^2\int\sigma d\Omega = -q\frac{1}{4\pi}a^2 2\pi a\left(1-\frac{y^2}{a^2}\right)\int_{-1}^{1}\frac{dx}{(y^2+a^2-2ayx)^{3/2}}, \text{ where } x = \cos\gamma$$

$$q_{induced} = -\frac{q}{2}a(a^2-y^2)\frac{2}{a(a^2-y^2)} = -q$$

c)

$$|F| = \left|\frac{qq'}{4\pi\varepsilon_0(y'-y)^2}\right| = \frac{1}{4\pi\varepsilon_0}\frac{q^2ay}{(a^2-y^2)},$$ the force is attractive, to the right.

d) If the conductor were fixed at a different potential, or equivalently if extra charge were put on the conductor, then the potential would be

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\vec{x}-\vec{y}|} + \frac{q'}{|\vec{x}-\vec{y}'|}\right] + V$$

and obviously the electric field in the sphere and induced charge on the inside of the sphere would remain unchanged.

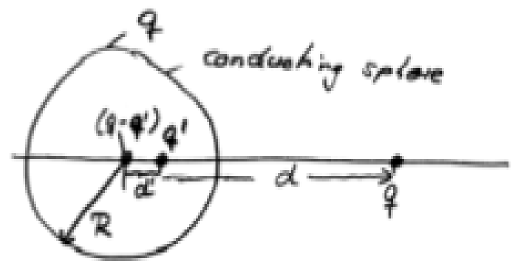
2) Jackson Problem 2.4 (see textbook chapter 2): *(40 points)*

$d > R$; q^+ on sphere

↳ q + image charge

$-q \cdot \frac{R}{d}$ at $d' = \frac{R^2}{d}$ +

charge $(q-q')$ at center



a.) $F = \frac{1}{4\pi\varepsilon_0}\left[\frac{q q'}{(d-d')^2} + \frac{q(q-q')}{d^2}\right] = \frac{1}{4\pi\varepsilon_0}\left[\frac{q\left(q + q\cdot\frac{R}{d}\right)}{d^2} + \frac{q\cdot\left(-q\, R/d\right)}{\left(d - R^2/d\right)^2}\right]$

$= \frac{q^2}{4\pi\varepsilon_0}\left[\frac{1 + R/d}{d^2} - \frac{R/d}{\left(d - \frac{R^2}{d}\right)^2}\right] = \frac{q^2}{4\pi\varepsilon_0 d^2}\left[1 + \frac{R}{d} - \frac{R/d}{\left(1 - (R/d)^2\right)^2}\right]$

use $x = R/d$ to get

$$F = \frac{q^2}{4\pi\varepsilon_0 d^2}\left[1 - x - \frac{x}{(1-x^2)^2}\right]$$

For $F = 0 \rightsquigarrow 1 + x - \frac{x}{(1-x^2)^2} = 0$

$\rightsquigarrow x \approx 0.61803$, $\frac{d}{R} = \frac{1}{x} = 1.617$

b) $a = d - R \ll 1$; $x = \frac{R}{d} = \frac{R}{R+a} \approx 1 - \frac{a}{R} + \frac{1}{2}\left(\frac{a}{R}\right)^2 + \dots$

↳ $F = \frac{q^2}{4\pi\varepsilon_0 d^2}\cdot\left[1 + x - \frac{x}{(1-x^2)(1-x^2)}\right] \approx \frac{q^2}{4\pi\varepsilon_0 d^2}\left[1 + 1 - \frac{a}{R} + \frac{1}{2}\left(\frac{a}{R}\right)^2 - \frac{1 - \frac{a}{R}}{\left(\frac{a}{R}\right)^2 \cdot 4}\right]$

$\approx \frac{q^2}{16\pi\varepsilon_0 a^2}$ = image force for a plane

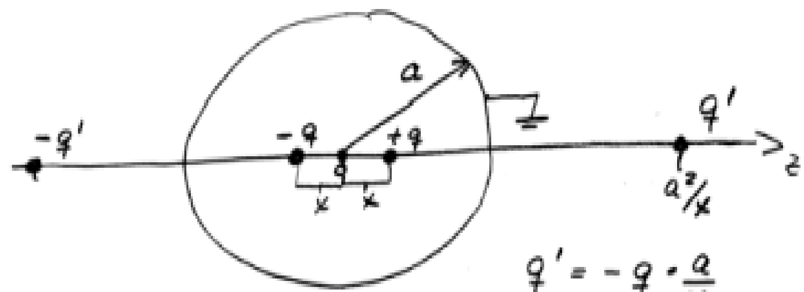
c.) solution is similar to (a)

$Q = 2q \rightsquigarrow d/R - 1 = 0.4276$

$Q = q/2 \rightsquigarrow d/R - 1 = 0.8823$

b.) - the same (charge on sphere is unimportant).

3) Using the method of images, (a) find the electric potential inside a grounded sphere due to a dipole at the center of the sphere, and (b) find the surface charge density on the sphere. *(30 points)*



$q' = -q \cdot \frac{a}{x}$

Potential inside the sphere is due to dipol $\vec{p}$ and charge induced on the sphere!

1) Consider the image charge of $-q$ and q $(\vec{p} = 2x \cdot q)$

$\rightsquigarrow$ $q = \frac{\vec{p}}{2x}$; $q' = -q\frac{a}{x}$ at $\frac{a^2}{x}$ and $-q' = q\frac{a}{x}$ @ $-\frac{a^2}{x}$

The electric field inside the sphere due to q' and $-q'$ when $x \to 0$:

$$E_1 = \frac{+q'}{4\pi\varepsilon_0 (a^2/x)^2} \cdot 2 = \frac{q \cdot a/x \cdot 2}{4\pi\varepsilon_0 (a^2/x)^2} = \frac{\frac{\vec{p}}{2x} \cdot \frac{a}{x} \cdot 2}{4\pi\varepsilon_0 (a^2/x)^2} = \frac{\vec{p}}{4\pi\varepsilon_0 a^3}$$

$\hookrightarrow E_1 = \text{const.} \rightsquigarrow \varphi_1 = -\int E_1 \, d\vec{z} = \frac{-\vec{p}}{4\pi\varepsilon_0 a^3} \cdot z = \underline{-\frac{p \cdot r \cdot \cos\theta}{4\pi\varepsilon_0 a^3}}$

2) Potential due to dipol $\vec{p}$:

$$\varphi_2 = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p} \cdot \vec{r}}{r^3} = \frac{1}{4\pi\varepsilon_0} \frac{p \cdot \cos\theta}{r^2}$$

3.) total potential $\varphi = \varphi_1 + \varphi_2 = \underline{\underline{\frac{p \cdot \cos\theta}{4\pi\varepsilon_0}\left(\frac{1}{r^2} - \frac{r}{a^3}\right)}}$

4.) Induced charged:

$$\sigma = +\varepsilon_0 \frac{\partial\varphi}{\partial r}\bigg|_a = \frac{p \cdot \cos\theta}{4\pi\varepsilon_0}\left(-\frac{2}{r^3} - \frac{1}{a^3}\right)\bigg|_{r=a} = \underline{\underline{-\frac{3 \cdot p \cdot \cos\theta}{4\pi\varepsilon_0 a^3}}}$$