Solutions for HW # 3

1) Problem 1.4, Jackson textbook:

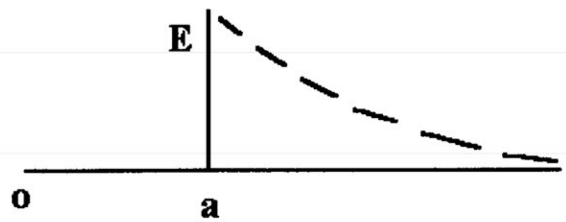
Gauss's Law:

$$\int \vec{E} \cdot d\vec{a} = \frac{Q_{enclosed}}{\varepsilon_0}$$

a) Conducting sphere: all of the charge is on the surface $\sigma = \frac{Q}{4\pi a^2}$

$$E4\pi r^2 = 0,\ r < a \qquad E4\pi r^2 = \frac{Q}{\varepsilon_0},\ r > a$$

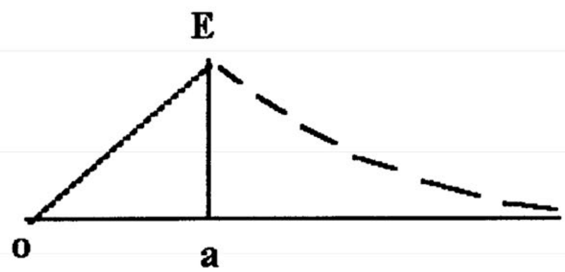
$$E = 0,\ r < a \qquad \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r},\quad r > a$$



b) Uniform charge density: $\rho = \frac{Q}{\frac{4}{3}\pi a^3}$, $r < a$, $\rho = 0$, $r > a$.

$$E4\pi r^2 = \frac{Qr^3}{\varepsilon_0 a^3} \rightarrow E = \frac{Qr}{4\pi\varepsilon_0 a^3},\quad r < a$$

$$E4\pi r^2 = \frac{Q}{\varepsilon_0} \rightarrow E = \frac{Q}{4\pi\varepsilon_0 r^2},\quad r > a$$



c) $\rho = A\, r^n$

$$Q = 4\pi\int r^2 dr A r^n = 4\pi A a^{n+3}/(n+3) \rightarrow \rho = \frac{(n+3)Q}{4\pi a^{n+3}} r^n,\ r < a$$

$$\rho = 0,\quad r > a$$

$$E4\pi r^2 = \frac{(n+3)Q}{4\pi\varepsilon_0 a^{n+3}} 4\pi r^{n+3}/(n+3) \rightarrow E = \frac{Q}{4\pi\varepsilon_0 r^2}\left(\frac{r^{n+3}}{a^{n+3}}\right),\quad r < a$$

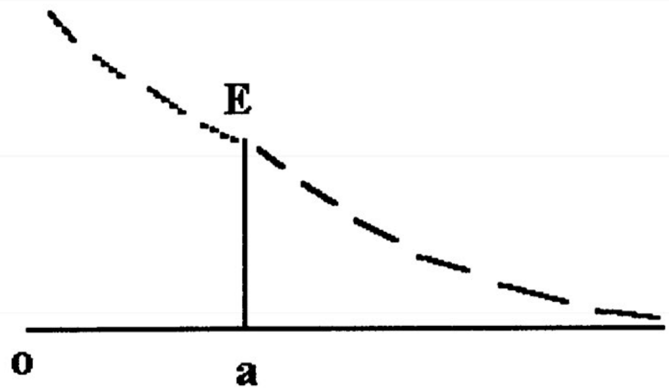
$$E = \frac{Q}{4\pi\varepsilon_0 r^2},\quad r > a$$

1. $n = -2$.

$$E = \frac{Q}{4\pi\varepsilon_0 r a},\quad r < a$$

1) Problem 1.4 - continued:

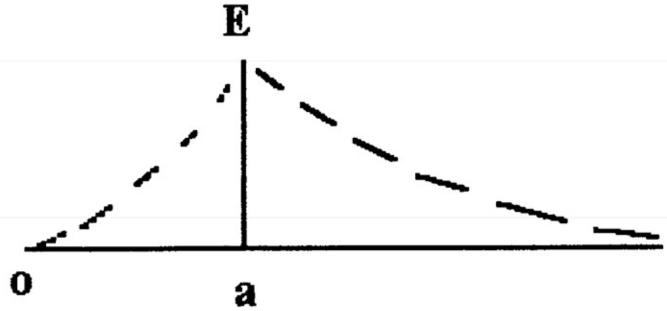
$$E = \frac{Q}{4\pi\varepsilon_0 r^2}, \quad r > a$$



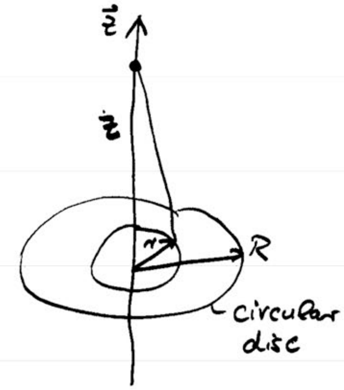
2. $n = 2$.

$$E = \frac{Qr^3}{4\pi\varepsilon_0 a^5}, \quad r < a$$

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}, \quad r > a$$



Problem 2) Find the potential and the electric field strength along the axis of a thin uniformly charged circular disc of radius *R* and total charge *q*. …..



1.) $$\varphi(z) = \frac{1}{4\pi\varepsilon_0} \int_S \frac{d^2\vec{r}\,'\ \sigma(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$$

$$= \frac{\sigma}{4\pi\varepsilon} \int_S \frac{d^2\vec{r}\,'}{|\vec{r} - \vec{r}\,'|}$$

$$= \frac{\sigma}{4\pi\varepsilon} \int_S \frac{d^2\vec{r}\,'}{\sqrt{z^2 + r^2}} = \frac{\sigma}{4\pi\varepsilon} \int_S \frac{2\pi r\, dr}{\sqrt{z^2 + r^2}}$$

$$= \frac{\sigma}{2\varepsilon} \int_0^R \frac{r\, dr}{\sqrt{z^2 + r^2}} = \frac{\sigma}{2\varepsilon} \sqrt{z^2 + r^2}\Big|_0^R = \frac{\sigma}{2\varepsilon}\left[\sqrt{z^2 + R^2} - \sqrt{z^2}\right]$$

$$\hookrightarrow \varphi(z) = \begin{cases} \sigma/2\varepsilon \cdot \left(\sqrt{z^2 + R^2} - z\right) & \text{for } z > 0 \\ \sigma/2\varepsilon \cdot \left(\sqrt{z^2 + R^2} + z\right) & \text{for } z < 0 \end{cases}$$

2.) From symmetry: $E_\varphi = E_r = 0$ on axis

$$\hookrightarrow E_z = \frac{\partial \varphi}{\partial z} = \begin{cases} \sigma/2\varepsilon \left(1 - z/\sqrt{z^2 + a^2}\right) & \text{for } z > 0 \\ \sigma/2\varepsilon \left(1 + z/\sqrt{z^2 + a^2}\right) & \text{for } z < 0 \end{cases}$$

3.) for $z \to +\infty$ $$E_z \to \frac{\sigma}{2\varepsilon}\left(1 - \left(1 + \left(\frac{a}{z}\right)^2\right)^{-1/2}\right) \approx \frac{\sigma}{2\varepsilon}\left(1 - 1 + \frac{1}{2}\frac{a^2}{z^2}\right)$$

$$= \frac{\sigma a^2}{4\varepsilon z^2} = q/4\pi\varepsilon z^2 = \text{field of point charge}$$

Similar $z \to -\infty$ $E_z \to -q/4\pi\varepsilon z^2$

4.) $$E(0^+) - E(0^-) = \frac{\sigma}{2\varepsilon} - \left(-\sigma/2\varepsilon\right) = \sigma/\varepsilon$$

5.) near disk: $$E_z\big|_{z\to 0^+} = \left(\frac{\sigma}{2\varepsilon}\cdot 1 - \frac{z}{\sqrt{z^2 + a^2}}\right)\Big|_{z\to 0} = \frac{\sigma}{2\varepsilon}$$ - field of infinitive plane.

Problem 3) Consider a spherically symmetric charge distribution $\rho=\rho(r)$. By dividing the charge distribution into spherical shells, find the potential and the electric field …

a.) Due to the spherical symmetry of the problem, we have

$$\vec{E} = E_r \cdot \hat{r}\,, \quad E_\theta = E_\varphi = 0$$

$$E_r = E_r(r)$$

with Gaussian surface: - sphere of r'

a1.) $\underline{r < r'}$: Gauss's law: $\oint_S \vec{E} \cdot d^2\vec{r} = q/\varepsilon_0$

$q = 0$ (charge is outside sphere)

$$\hookrightarrow \oint_S \vec{E}\, d^2\vec{r}\,' = \oint_S E_r\, d^2 r' = E_r(r) \cdot 4\pi r^2 = 0 \Rightarrow \underline{\underline{E_r(r) = 0}}$$

a2.) $\underline{r > r'}$: $\oint_S \vec{E}\, d^2\vec{r}\,' = E(r) \cdot 4\pi r^2 = q/\varepsilon = \dfrac{\sigma \cdot 4\pi r'^2}{\varepsilon}$

$$\hookrightarrow E_r(r) = \frac{\sigma}{\varepsilon}\left(\frac{r'}{r}\right)^2 \Rightarrow E_r(r) = \begin{cases} 0 & \text{for } r < r' \\ \sigma/\varepsilon \cdot (r'/r)^2 & \text{for } r > r' \end{cases}$$

b.) potential:

$$\varphi(r) = \varphi(\infty) + \int_r^\infty E(r)\, dr$$

b1: $\underline{r > r'}$: $\rightsquigarrow \varphi(r) = \int_r^\infty \frac{\sigma}{\varepsilon}\left(\frac{r'}{r}\right)^2 dr = \sigma/\varepsilon\, (r')^2 \cdot \frac{1}{r}\Big|_r^\infty = \underline{\underline{\frac{\sigma}{\varepsilon}\frac{(r')^2}{r}}}$

b2: $\underline{r < r'}$: $\varphi(r) = \int_r^\infty E(r)\, dr = \underbrace{\int_r^{r'} E(r)\, dr}_{=0} + \int_{r'}^\infty E(r)\, dr = \frac{\sigma}{\varepsilon}\frac{(r')^2}{r'} = \underline{\underline{\frac{\sigma \cdot r'}{\varepsilon}}}$

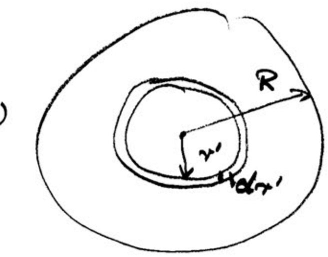
$$\hookrightarrow \varphi = \begin{cases} \sigma/\varepsilon\, (r')^2/r & \text{for } r > r' \\ \dfrac{\sigma \cdot r'}{\varepsilon} & \text{for } r < r' \end{cases}$$

c.) Next, a spherically symmetric charge distribution $\rho(r)$ of Radius R can be represented as a sum of infinitely thin spherical shells of radius r' ($0 \le r' \le R$) and $\Rightarrow$

Problem 3) continued …

→ surface charge density

$$\sigma = \frac{4\pi (r')^2\, dr'\, \rho(r')}{4\pi (r')^2} = \rho(r')\, dr' = \sigma(r')$$



Due to superposition principle, the total electric field and potential are calculated as integrals over $\underline{r'}$.

1.) $\underline{r > R}$: $E_r(r) = \int_0^R dE_r(r')$, where $dE_r(r')$ is the field created by spherical shell of radius r' at point r and thickness dr'.

$$\hookrightarrow E_r(r) = \int_0^R \frac{\sigma(r')}{\varepsilon}\left(\frac{r'}{r}\right)^2 dr' = \int_0^R \frac{1}{\varepsilon}\rho(r')\frac{r'^2}{r^2}\,dr' = \frac{1}{\varepsilon_0 r^2}\int_0^R \rho(r')\, r'^2\, dr'$$

$$\text{potential } \varphi(r) = \int_0^R d\varphi(r') = \int_0^R \frac{\sigma(r')}{\varepsilon}\frac{(r')^2}{r} = \int_0^R \frac{1}{\varepsilon_0 r}\rho(r')(r')^2\, dr'$$

Find the surface charge densities on the plates.

2.) $\underline{r < R}$:

$$E_r(r) = \int_0^R dE_r(r') = \int_0^r \frac{1}{\varepsilon}\rho(r')\left(\frac{r'}{r}\right)^2 dr' = \frac{1}{\varepsilon r^2}\int_0^r \rho(r')\, r'^2\, dr'$$

$$\text{and } \varphi(r) = \int_0^R d\varphi(r') = \underbrace{\int_0^r d\varphi(r')}_{r' < r} + \underbrace{\int_r^R d\varphi(r')}_{(r' > r)} = \int_0^r \frac{\sigma(r')\, r'^2}{\varepsilon \cdot r} + \int_r^R \frac{\sigma(r')\, r'}{\varepsilon}$$

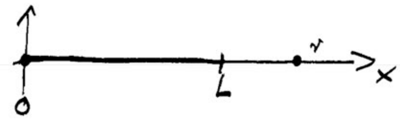
$$= \frac{1}{\varepsilon_0 r}\int_0^r \rho(r')\cdot(r')^2\, dr' + \frac{1}{\varepsilon}\int_r^R \rho(r')\, r'\, dr'$$

$$\Rightarrow \quad r > R: \quad E_r(r) = \frac{1}{\varepsilon r^2}\int_0^R \rho(r')\, r'^2\, dr', \quad \varphi(r) = \frac{1}{\varepsilon r}\int_0^R \rho(r')\, r'^2\, dr'$$

$$r < R: \quad E_r(r) = \frac{1}{\varepsilon r^2}\int_0^r \rho(r')\, r'^2\, dr', \quad \varphi(r) = \frac{1}{\varepsilon r}\int_0^r \rho(r')\, r'^2\, dr' + \frac{1}{\varepsilon}\int_r^R \rho(r')\, r'\, dr'$$

Problem 4) A line conductor of length L and total charge Q lies on the x-axis with one end on the origin. Find the electric potential and the electric field …

$\rho(x) = \frac{Q}{L} \quad (0 \le x \le L)$



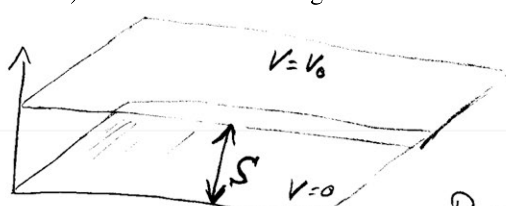
potential $\psi(x) = \frac{1}{4\pi\varepsilon} \int_0^L \frac{\rho(x')\, d^3x'}{|\vec{x}-\vec{x}'|} = \frac{1}{4\pi\varepsilon} \frac{Q}{L} \int_0^L \frac{dx'}{|x-x'|}$

$$= -\frac{1}{4\pi\varepsilon} \frac{Q}{L} \ln(x-x')\Big|_0^L = -\frac{1}{4\pi\varepsilon} \frac{Q}{L} \ln\left(\frac{x-L}{x}\right)$$

$$= -\frac{1}{4\pi\varepsilon} \frac{Q}{L} \cdot \ln\left(1-\frac{L}{x}\right)$$

field $\vec{E}(\vec{x}) = -\nabla\psi(x) = -\frac{1}{4\pi\varepsilon} \frac{Q}{L} \frac{1}{\left(1-\frac{L}{x}\right)} \cdot \left(\frac{L}{x^2}\right) = \underline{\frac{1}{4\pi\varepsilon} \cdot \frac{Q}{(x-L)\,x}}$

Problem 5) Two infinite parallel plates separated by a distance s are at the potentials 'zero' and V_0. a) find the potential … and b) Find the surface charge densities on the plates.



$\rho = \rho_0 (x/s)$

Poisson Equ. $\Delta\psi = -\rho/\varepsilon$

a.) $\frac{\partial^2\psi}{\partial x^2} = \frac{-\rho_0(x/s)}{\varepsilon_0}$; $\rightsquigarrow \frac{\partial\psi}{\partial x} = -\rho_0/\varepsilon_0 \cdot \frac{x^2}{2s} + C_1$

$\hookrightarrow \psi = -\rho_0/\varepsilon_0 \cdot x^3/6s + C_1 x + C_2$

boundary: $\psi(0) = 0 = C_2$

$\psi(s) = V_0 = -\rho_0/\varepsilon_0 \cdot \frac{s^3}{6s} + C_1 \cdot s \rightsquigarrow C_1 = \frac{1}{s}\left(V_0 - \frac{\rho_0}{\varepsilon_0}\frac{s^2}{6}\right)$

$$\hookrightarrow \boxed{\psi(x) = -\frac{\rho_0}{\varepsilon_0} \cdot \frac{x^3}{6s} + \left(\frac{V_0}{s} + \frac{\rho_0}{\varepsilon_0} \cdot \frac{s}{6}\right) \cdot x}$$

b.) $\vec{E} = -\nabla\psi$; $\vec{E} = -\frac{\partial}{\partial x}\left(-\frac{\rho_0}{\varepsilon_0}\frac{x^3}{6s} + \left(\frac{V_0}{s} + \frac{\rho_0 \cdot s}{6\varepsilon_0}\right) x\right)$

$$= -\frac{\rho_0}{\varepsilon_0}\frac{x^2}{2s} + \frac{V_0}{s} + \frac{\rho_0 \cdot s}{6 \cdot \varepsilon_0}$$

for $x=0$: $E_x = \frac{\sigma}{\varepsilon_0} \rightsquigarrow \sigma = \frac{-\rho_0 x^2}{2s} + \frac{V_0 \cdot \varepsilon_0}{s} + \frac{\rho_0 \cdot s}{6\varepsilon_0}$, for $x=s$: $E_x = -\frac{\sigma}{\varepsilon_0}$