Solutions for Homework # 2

Problem#1: (Jackson 1.1)

1 *a)* First consider a closed hollow conductor with interior charges. Total charge on the original conductor is Q; charge place in the interior is q.
Take a Gaussian surface just outside the surface. Then

$$\oint \vec{E}\cdot\hat{n} = \frac{1}{\varepsilon_o}\cdot\int_V d^3r\cdot\rho(\vec{r}). \qquad \text{So,} \quad E_\perp\cdot A = \frac{Q+q}{\varepsilon_o} \quad \text{or} \quad E_\perp\cdot A = \frac{Q+q}{A\cdot\varepsilon_o}$$

And $E_\perp \neq 0$, so the outside is not protected from those interior charges. Now consider a conductor with external charges. Take a path from point a on the surface to point b on the surface straight through the conductor. Then use:

$$\oint \vec{E}\cdot d\vec{l} = 0 \qquad \text{So,} \qquad \int_{\mathit{surface\text{-}path}} \vec{E}\cdot d\vec{l} + \int_{\text{interior-}\mathit{path}} \vec{E}\cdot d\vec{l} = 0$$

But the integral over the surface part of the path is zero, since there is no flied arbitrarily close to the surface. Then

$$\int_{\text{interior-}\mathit{path}} \vec{E}\cdot d\vec{l} = 0$$

But this means that, if $\vec{E}_{\text{interior-}\mathit{path}} \neq 0, \qquad V_a \neq V_b$

However, a conductor is an equipotential. Therefore, the electric field on the interior of the sphere must be zero.

b) Consider a conductor and a path enclosing a bit of the surface. Now we know that:

$$\oint \vec{E}\cdot d\vec{l} = 0$$

Now, squish the side of the path which are perpendicular to the surface of the conductor; then their contribution to above integral is zero. So,

$$\oint \vec{E}\cdot d\vec{l} = \int_{\mathit{inside}} \vec{E}\cdot d\vec{l} - \int_{\mathit{outside}} \vec{E}\cdot d\vec{l}\,.$$

But $\vec{E} = 0$ inside the conductor, so $\oint \vec{E}\cdot d\vec{l} = \int_{\mathit{outside}} \vec{E}\cdot d\vec{l} = 0$ or $E_{\perp\text{outside}}\cdot l = 0$

Since l is arbitrary, $E_{\perp\text{outside}} = 0$.

Therefore, the electric field just outside of a conductor is perpendicular to the surface of the conductor. Now take a Gaussian pill box enclosing the surface. Squish the side walls, so that they are infinitesimal. Then,

$$\oint_S \vec{E}\cdot\hat{n} = \frac{1}{\varepsilon_o}\cdot\int_V d^3r\cdot\rho(\vec{r}) \quad \text{can be written as} \quad \oint_{\mathit{inside}} \vec{E}\cdot d\hat{n} + \oint_{\mathit{outside}} \vec{E}\cdot d\hat{n} = \frac{1}{\varepsilon_o}\cdot\int_V d^3r\cdot\rho(\vec{r})$$

But $E_{\mathit{inside}} = 0$. Now squish the box so that $\vec{E} \sim$ constant over A. Then,

$$\oint_{\mathit{outside}} \vec{E}\cdot d\hat{n} = \frac{A\cdot\sigma}{\varepsilon_o}$$

$$\text{or} \qquad E_{\mathit{outside}}\cdot A = \frac{A\cdot\sigma}{\varepsilon_o} \qquad \Rightarrow \qquad E_{\mathit{outside}} = \frac{\sigma}{\varepsilon_o}$$

Problem#2: Jackson 1.2 - textbook page 51

Generalize $\mathcal{D}(\alpha, x, y, z)$ as 3-dim. delta function of the form

$$\delta^{(3)}(\vec{r}-\vec{r}_0) = \frac{1}{(\alpha \cdot \sqrt{2\pi})^3} \cdot \exp\left[-\frac{1}{2\alpha^3}\left((x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right)\right]$$

$$= \delta(x-x_0)\cdot\delta(y-y_0)\cdot\delta(z-z_0)$$

with the property: $\int d^3r \; F(x,y,z)\, \delta^{(3)}(\vec{r}-\vec{r}_0) = F(x_0, y_0, z_0)$

Now assume $dx = du/U$, $dy = dv/V$, $dz = dw/W$, $d^3r = du\,dv\,dw / U\cdot V\cdot W$

$\hookrightarrow \; (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = \frac{(u-u_0)^2}{U^2} + \frac{(v-v_0)^2}{V^2} + \frac{(w-w_0)^2}{W}$

For $\alpha \to 0$: u, v and w $\to u_0, v_0, w_0$

which means we can interchange (x, y, z) with (u, v, w) by scaling (see notes)

$$F(x_0, y_0, z_0) = \int \frac{du\,dv\,dw}{U\cdot V\cdot W} \; F(u,v,w) \cdot \exp\left[-\frac{1}{2\alpha^3}\left\{\frac{(u-u_0)^2}{U^2} + \frac{(v-v_0)^2}{V^2} + \frac{(w-w_0)^2}{W^2}\right\}\right]$$

$$= \int F(u,v,w) \cdot \frac{du}{U} \cdot e^{-1/2\alpha^3 \cdot \frac{(u-u_0)^2}{U^2}} \cdot \frac{dv}{V} e^{-1/2\alpha^3 \frac{(v-v_0)^2}{V^2}} \cdot \frac{dw}{W} e^{-1/2\alpha^3 \frac{(w-w_0)^2}{W}}$$

$$= \int \frac{d^3r}{(\sqrt{2\pi}\cdot\alpha)^3} \cdot F(x,y,z) \; e^{-\frac{1}{2\alpha^3}\cdot(x-x_0)^2} \cdot e^{-\frac{1}{2\alpha^3}\cdot(y-y_0)^2} \cdot e^{-\frac{1}{2\alpha^3}\cdot(z-z_0)^2}$$

with volume element $d^3r = dx\,dy\,dz = \frac{1}{U\cdot V\cdot W} \cdot du\,dv\,dw$

compare δ-function:

$\hookrightarrow \; \delta(r-r_0) = \delta(u-u_0)\,\delta(v-v_0)\,\delta(w-w_0) \cdot (U\cdot V\cdot W)$

Problem#3: Jackson 1.5 - textbook page 51

$$\phi(r) = \frac{q\, e^{-\alpha \cdot r}}{4\pi \varepsilon_0 \cdot r}\left(1 + \frac{\alpha r}{2}\right)$$

Possion's Equ. $\nabla^2 \phi(r) = -\frac{\rho(r)}{\varepsilon_0}$, $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right)$

$$\nabla^2 \phi = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}\left(\frac{e^{-\alpha r}}{r} + \frac{\alpha}{2} e^{-\alpha \cdot r}\right)\right]$$

Using $\nabla^2 \frac{1}{r} = -4\pi \delta(\vec{r})$ ← point charge

$$+ \quad \nabla^2 \phi = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r} e^{-\alpha r}\left(\frac{1}{r} + \frac{\alpha}{2}\right)\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left[e^{-\alpha r}\left(-\alpha \cdot r - \frac{r^2\alpha^2}{2}\right) + r^2 e^{-\alpha r}\left(\frac{1}{-r^2}\right)\right]$$

$$\left[e^{-\alpha r}\left(-1 - \alpha r - \frac{r^2\alpha^2}{2}\right)\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{e^{-\alpha \cdot r}}{r^2}\left(\alpha + \alpha^2 r + \frac{r^2\alpha^3}{2}\right) + \frac{e^{-\alpha r}}{r^2}\left(-\alpha - \alpha^2 r\right)\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{e^{-\alpha \cdot r}}{r^2}\left(\frac{r^2\alpha^2}{2}\right) = \frac{q}{8\pi\varepsilon_0}\alpha^3 e^{-\alpha \cdot r}$$

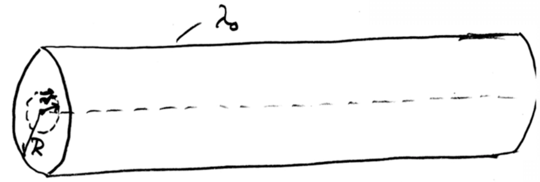
$$\rho(\vec{r}) = -\frac{q}{4\pi}\nabla^2\left(\frac{1}{r}\right) - \varepsilon_0 \nabla^2 \phi$$

$$= q\,\delta(r) - \frac{q}{8\pi}\alpha^3 e^{-\alpha r}$$

That is, the charge distribution consist of a positive point charge at the origin, plus an exp.-decreasing negatively charged cloud.

Problem#4: An infinitely long cylinder of radius R has a line charge density. Find the electric field and its scalar potential λ_o everywhere …..

a.)



$$\int_S d^2r\, \vec{E}\cdot d\hat{n} = \frac{1}{\epsilon_0}\int_V d^3r\, \rho(r)$$

↓ cylindrical coord.

$$2\pi\rho\cdot L\cdot E_\rho = \frac{1}{\epsilon_0}\lambda_0\cdot L, \quad \rho < R$$

For a conductor, all charges are on the outer surface.

For $\rho < R \rightsquigarrow E = 0$; from $E = -\nabla\phi$, $\phi = -\int_0^r \vec{E}\, dr = 0$

↳ ϕ = const, let's take $\phi = 0$ everywhere inside of cylinder

For $\underline{\rho > R}$: $E_\rho = \frac{\lambda_0}{2\pi\epsilon_0\rho}\cdot\hat{\rho}$

$$\rightsquigarrow \phi(\rho) = -\int_R^\rho \vec{E}\, d\vec{r} = -\frac{\lambda_0}{2\pi\epsilon_0}\int_R^\rho \frac{1}{\rho'}\, d\rho' = -\frac{\lambda_0}{2\pi\epsilon_0}\ln(\rho/R)$$

b.) $\rho(\vec{r}) = \frac{\lambda_0}{\pi R^2}$ for $r \le R$

$= 0$ for $r > R$

$r < R$: $E_r\cdot 2\pi r = \frac{1}{\epsilon_0}\int \rho(\vec{r})\cdot\rho\, d\hat{\varphi}\, d\varphi\, dz = \frac{1}{\epsilon_0}\int\frac{\lambda_0}{\pi R^2}\rho\, d\varphi\, d\varphi\, dz$

$$= \frac{\lambda_0}{\epsilon_0 R^2}\int_0^r \rho\, d\varphi = \frac{\lambda_0}{\epsilon_0 R^2} r^2$$

$$\rightsquigarrow E_r = \frac{\lambda_0\cdot r}{2\pi\epsilon_0 R^2}\cdot\hat{r}$$

$r > R$: $\vec{E} = \frac{\lambda_0}{2\pi\epsilon_0\cdot r}$, with $\phi(a) = 0$ $(a < R)$

$$\rightsquigarrow \int_a^r \vec{E}\cdot d\vec{r} = -\phi(r) + \underbrace{\phi(a)}_{=0}$$

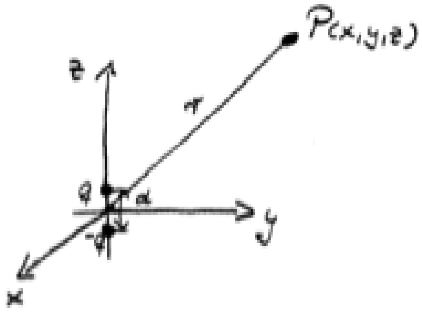
For $r < R$: $\int_a^r \frac{\lambda_0}{2\pi\epsilon_0 R^2}\, r\, dr = \frac{\lambda_0}{2\pi\epsilon_0 R^2}\left(\frac{r^2}{2}\right)\Big|_a^r$

$$\rightsquigarrow \phi(r) = -\frac{\lambda_0}{4\pi\epsilon_0 R^2}(r^2 - a^2)$$

for $r > R$: $\phi(r) = -\int_a^r \frac{\lambda_0}{2\pi\epsilon_0 r}\, dr = -\frac{\lambda_0}{2\pi\epsilon_0}\ln\left(\frac{r}{a}\right)$

Problem#5: The electric potential of a dipole $\vec{P}$ at origin is …..

a.) Scalar potential of dipol @ origin



$$\phi = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot\vec{x}}{r^3}$$

$\vec{p}$: dipol moment along $\hat{z}$-axis
(from $-q \rightarrow +q$ with distance d)

$|\vec{p}| = q\cdot d$, $\vec{p} = p_x\cdot\hat{x} + p_y\cdot\hat{y} + p_z\cdot\hat{z}$
$\vec{x} = x\cdot\hat{x} + y\cdot\hat{y} + z\cdot\hat{z}$ (Position vector)

$\hookrightarrow \vec{p}\cdot\vec{x} = p_x\cdot x + p_y\cdot y + p_z\cdot z$

$$\vec{E} = -\nabla\phi = -\left[\frac{\partial\phi}{\partial x}\cdot\hat{x} + \frac{\partial\phi}{\partial y}\cdot\hat{y} + \frac{\partial\phi}{\partial z}\cdot\hat{z}\right]$$

$$\frac{\partial\phi}{\partial x} = \frac{1}{4\pi\epsilon_0}\frac{\partial}{\partial x}\left[\frac{p_x\cdot x + p_y\cdot y + p_z\cdot z}{(x^2+y^2+z^2)^{3/2}}\right] = \dots = \frac{1}{4\pi\epsilon_0} p_x\cdot\left[\frac{1}{r^3} - \frac{3x^2}{r^5}\right]$$

similarly $\partial\phi/\partial y$ and $\partial\phi/\partial z$

$$\hookrightarrow \vec{E} = -\nabla\phi = \frac{-\vec{p}}{4\pi\epsilon_0} + \frac{3}{4\pi\epsilon_0\cdot r^5}\cdot\left[x^2 p_x\hat{x} + y^2\cdot p_y\cdot\hat{y} + z^2\cdot p_z\cdot\hat{z}\right]$$

b.) $\underline{\vec{p} \parallel z\text{-axis}}$: $\vec{p} = p\cdot\hat{z} \rightsquigarrow \vec{p}\cdot\vec{x} = p\,\hat{z}\cdot[x\cdot\hat{x} + y\cdot\hat{y} + z\cdot\hat{z}] = p\cdot z$

$$\hookrightarrow \phi = \frac{1}{4\pi\epsilon_0}\frac{p\cdot z}{r^3} \underset{\text{spherical}}{=} \frac{1}{4\pi\epsilon_0}\cdot\frac{p\cdot\cos\theta}{r^2}, \quad (z = r\cos\theta!)$$

$$\text{with } \nabla\phi = \hat{r}\,\partial\phi/\partial r + \hat{\theta}\frac{1}{r}\frac{\partial\phi}{\partial\theta} + \hat{\varphi}\cdot\frac{1}{r\cdot\sin\theta}\,\partial\phi/\partial\varphi$$

$$\hookrightarrow E_r = \frac{p\cdot\cos\theta}{2\pi\epsilon_0 r^3}, \quad E_\theta = \frac{p\cdot\sin\theta}{4\pi\epsilon_0 r^3}, \quad E_\varphi = 0$$