Solutions for Homework # 1

Problems #1-3:

① $$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{A}\cdot\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix} = A_x(B_y C_z - C_y B_z) + A_y(C_x B_z - C_z B_x) + A_z(B_x C_y - B_y C_x) \quad (1)$$

$$(\vec{A}\times\vec{B})\cdot\vec{C} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}\cdot\vec{C} = C_x(A_y B_z - A_z B_y) + C_y(A_z B_x - A_x B_z) + C_z(A_x B_y - A_y B_x) \quad (2)$$

Comparing (1) and (2) $\rightsquigarrow$ $\vec{A}(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$

② $$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) \; ?$$

$$\vec{A}\times\vec{B}\times\vec{C} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ (B_y C_z - B_z C_y) & (C_x B_z - C_z B_x) & (B_x C_y - B_y C_x)\end{vmatrix} = \dots$$

$$= \begin{pmatrix} A_y(B_x C_y - B_y C_x) - A_z(C_x B_z - C_z B_x) + A_x B_x C_x - A_x B_x C_x \\ \dots \\ \dots \end{pmatrix}$$

$$= \begin{pmatrix} B_x\cdot(A_x C_x + A_y C_y + A_z C_z) - C_x\cdot(A_x B_x + A_y B_y + A_z B_z) \\ \dots \\ \dots \end{pmatrix}$$

$$= \vec{B}\cdot(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

③ $$\nabla\times\nabla u = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial_x & \partial_y & \partial_z\\ \partial_x u & \partial_y u & \partial_z u\end{vmatrix} = \hat{i}\left(\frac{\partial^2 u}{\partial y\partial z} - \frac{\partial^2 u}{\partial z\partial y}\right) + \hat{j}\left(\frac{\partial^2 u}{\partial x\partial z} - \frac{\partial^2 u}{\partial z\partial x}\right) + \hat{k}\left(\frac{\partial^2 u}{\partial y\partial x} - \frac{\partial^2 u}{\partial x\partial y}\right)$$

$$= 0$$

$\rightarrow$

Problems #4 - 9:

④ $\nabla \cdot (\nabla \times \vec{A}) =$

$$\vec{\nabla} \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} = \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} + \frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial y \partial x} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y} = \underline{0}$$

⑤ $\frac{d}{d\sigma}(u\vec{A}) = \ldots \rightsquigarrow \left[\frac{d}{d\sigma}(u\vec{A})\right]_x = \frac{d}{d\sigma}(uA_x) = \underline{\frac{du}{d\sigma}A_x + u\frac{dA_x}{d\sigma}}$

similarly for y- and z- component

⑥ $\frac{d}{d\sigma}(\vec{A}\cdot\vec{B}) = \frac{d}{d\sigma}(A_xB_x + A_yB_y + A_zB_z) = \frac{dA_x}{d\sigma}\cdot B_x + A_x\frac{dB_x}{d\sigma} + \ldots = \underline{\frac{d\vec{A}}{d\sigma}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{d\sigma}}$

⑦ $\nabla(u+v) = \left(\frac{\partial(u+v)}{\partial x}, \frac{\partial(u+v)}{\partial y}, \frac{\partial(u+v)}{\partial z}\right) = \ldots = \underline{\nabla u + \nabla v}$

⑧ $\nabla(u\cdot v) = \left(\frac{\partial(u\cdot v)}{\partial x}, \frac{\partial(u\cdot v)}{\partial y}, \frac{\partial(u\cdot v)}{\partial z}\right)$

$$= \left(\frac{\partial u}{\partial x}\cdot v + u\frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}\cdot v + u\cdot\frac{\partial v}{\partial y}, \frac{\partial u}{\partial z}\cdot v + u\cdot\frac{\partial v}{\partial z}\right)$$

$$= \nabla u \cdot v + u\cdot\nabla v$$

⑨ $\nabla(\vec{c}\cdot\vec{r}) = ?$

$[\nabla(\vec{c}\cdot\vec{r})]_x = \frac{\partial}{\partial x}(\vec{c}\cdot\vec{r}) = \frac{\partial}{\partial x}(c_x\cdot x + c_y y + c_z\cdot z)$

$$= \frac{\partial c_x}{\partial x}\cdot x + c_x\cdot 1 + \frac{\partial c_y}{\partial x}\cdot y + \cancel{y\cdot\frac{\partial y}{\partial x}}^{=0} + \frac{\partial c_z}{\partial x}\cdot z + \cancel{c_z\frac{\partial z}{\partial x}}^{=0}$$

$= \ldots = (\nabla\vec{c})\cdot\vec{r} + \vec{c}$

if $\vec{c}$ = constant $\rightsquigarrow \nabla\vec{c} = 0 \Rightarrow \nabla(\vec{c}\cdot\vec{r}) = \underline{\vec{c}}$

Problems #10 - 14:

(10)
$$\nabla(\vec{A}+\vec{B}) = \frac{\partial}{\partial x}(A_x+B_x)+\frac{\partial}{\partial y}(A_y+B_y)+\frac{\partial}{\partial z}(A_z+B_z)$$
$$= \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}+\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z} = \underline{\underline{\vec{\nabla}\cdot\vec{A}+\vec{\nabla}\cdot\vec{B}}}$$

(11)
$$\vec{\nabla}\cdot(u\vec{A}) = \frac{\partial}{\partial x}(uA_x)+\frac{\partial}{\partial y}(u\cdot A_y)+\frac{\partial}{\partial z}(uA_z)$$
$$= \dots = \underline{\underline{\nabla u\cdot\vec{A}+u\cdot\nabla\cdot\vec{A}}}$$

(12)
$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \frac{\partial}{\partial x}(A_yB_z-A_zB_y)+\frac{\partial}{\partial y}(A_zB_x-A_xB_z)+\frac{\partial}{\partial z}(A_xB_y-A_yB_x)$$
$$= B_z\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)+B_y\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+B_x\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)$$
$$+A_y\left(\frac{\partial B_z}{\partial x}-\frac{\partial B_x}{\partial z}\right)+A_z\left(\frac{\partial B_x}{\partial y}-\frac{\partial B_y}{\partial x}\right)+A_x\left(\frac{\partial B_y}{\partial z}-\frac{\partial B_z}{\partial y}\right)$$
$$= \underline{\underline{\vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}(\nabla\times\vec{B})}}$$

(13)
$$(\nabla\times(u\vec{A}))_x = \frac{\partial(uA_z)}{\partial y}-\frac{\partial(uA_y)}{\partial z} = \underbrace{u\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)}_{=(\nabla\times A)_x\cdot u}+\underbrace{A_z\frac{\partial u}{\partial y}-A_y\frac{\partial u}{\partial z}}_{(\nabla u\times A)_x}$$
$$\rightsquigarrow \nabla\times(u\vec{A}) = \underline{\underline{u\cdot(\nabla\times\vec{A})+\nabla u\times\vec{A}}}$$

(14)
$$(\nabla\times(\nabla\times\vec{A}))_x = \frac{\partial}{\partial y}\left((\nabla\times\vec{A})_x\right)-\frac{\partial}{\partial z}\left((\nabla\times A)_y\right)$$
$$= \frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial A_y}{\partial y}\right)-\frac{\partial^2 A_x}{\partial y^2}-\frac{\partial^2 A_x}{\partial z^2}+\frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial z}\right)-\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_x}{\partial x^2}$$
$$= \frac{\partial}{\partial x}\underbrace{\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)}_{=(\nabla\cdot\vec{A})}-\underbrace{\left(\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_x}{\partial y^2}+\frac{\partial^2 A_x}{\partial z^2}\right)}_{=\Delta A_x}$$
$$\hookrightarrow \nabla\times(\nabla\times\vec{A}) = \underline{\underline{\nabla(\nabla\cdot\vec{A})-\Delta\vec{A}}}$$