Unstable periodic orbits and the natural measure of nonhyperbolic chaotic saddles

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Chaotic saddles are nonattracting dynamical invariant sets that physically lead to transient chaos. We examine the characterization of the natural measure by unstable periodic orbits for *nonhyperbolic* chaotic saddles in *dissipative* dynamical systems. In particular, we compare the natural measure obtained from a long trajectory on the chaotic saddle to that evaluated from unstable periodic orbits embedded in it. Our systematic computations indicate that the periodic-orbit theory of the natural measure, previously shown to be valid only for hyperbolic chaotic sets, is applicable to nonhyperbolic chaotic saddles as well. [S1063-651X(99)08311-7]

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Chaotic saddles are nonattracting dynamical invariant sets that arise in many situations of physical interest [1–3]. A trajectory starting from a random initial condition in a phase-space region containing a chaotic saddle typically stays near the saddle and exhibits a chaoticlike dynamics for a finite amount of time before asymptoting to a final state (usually not chaotic). Chaos in this case is only transient. Mathematically, a chaotic saddle is a closed, bounded, and invariant set with a dense orbit. Chaotic saddles are the *soul* of chaotic dynamics because they can be produced by the horseshoe-type dynamics [4]. It is known that chaotic saddles can lead to important physical phenomena such as chaotic scattering [5], fractal basin boundaries [6], fractal concentrations of passive particles advected in open hydrodynamical flows [7], and fractal distribution of chemicals in environmental flows [8]. It is therefore of paramount physical interest to be able to understand and characterize chaotic saddles in terms of the fundamental dynamical quantities. And there is nothing more fundamental than the infinite number of unstable periodic orbits [9] embedded in the saddle. In this regard, we note that the most physically relevant characteristic associated with a chaotic saddle is its natural measure because dynamical invariants of the saddle such as the Lyapunov exponents, the fractal dimension, and other averages of physical observables are meaningful only when the measure being considered is the natural one. Thus, it is of primary importance to quantify the natural measure of the chaotic saddle by the infinite set of unstable periodic orbits embedded in it.

In this paper, we focus on nonhyperbolic chaotic saddles arising in *dissipative* chaotic systems that can be described by two-dimensional noninvertible maps or, equivalently, three-dimensional flows [10]. For such systems, the source of nonhyperbolicity is the set of infinite numbers of tangency points between the stable and unstable manifolds [11]. As we will discuss later, in this case, if one distributes a larger number of initial conditions in a phase-space region containing the chaotic saddle, the number of chaotic trajectories in the region decreases *exponentially* in time [12]. There have been many papers addressing the role of nonhyperbolic chaotic saddles in chaotic dynamics [13–18]. The focus of this paper is on the period-orbit characterization of the natural measure of chaotic saddles. An important contribution in this direction was made by Grebogi, Ott, and Yorke [19], who obtained an expression for the invariant natural measure in terms of the magnitude of the eigenvalues of the unstable periodic orbits embedded in the chaotic attractor. They proved [19] the correctness of their expression but only for the special case of hyperbolic dynamics [11]. The validity of their results for physical, which are typically nonhyperbolic, situations remained, however, only a conjecture. Recently, it was numerically verified that this quantification of the natural measure by unstable periodic orbits was valid for nonhyperbolic chaotic attractors [20]. Since chaotic saddles give rise to very distinct physical phenomena from those by chaotic attractors, and since chaotic saddles can be commonly nonhyperbolic [21,22], it is important to assess the validity of the periodic-orbit characterization of the natural measure for nonhyperbolic chaotic saddles. The aim of this paper is then to provide an analysis and solid numerical evidence for such a characterization. We note that due to the nonattracting nature of chaotic saddles, the natural measure and its characterization by periodic orbits become highly nontrivial and more sophisticated compared with the case of chaotic attractors.

We begin by introducing the natural measure of a chaotic saddle and expressing it in terms of unstable periodic orbits. Consider dynamical systems described by two-dimensional invertible maps, $\mathbf{x}_{n+1}=\mathbf{M}(\mathbf{x}_n)$, where $\mathbf{x}\in\mathbf{R}^2$. These maps arise on the Poincaré surface of section of three-dimensional flows. Imagine a phase-space region S that contains a nonattracting chaotic saddle. The stable and the unstable manifolds of the chaotic saddle are sets of points that asymptote to the chaotic saddle under the forward and backward iterations of the map, respectively. If a large number N_0 of random initial conditions are distributed in S, the corresponding trajectories will leave S eventually as time progresses. They do so by being attracted along the stable manifold, wandering near the chaotic saddle, and then exiting along the unstable manifold. Let $N(n)$ be the number of trajectories that still remain in S at time n. For large n, this number $N(n)$ decreases exponentially due to the chaotic nature of the saddle in dissipative systems:

$$N(n)=N_0 e^{-n/\tau}, \tag{1}$$

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where τ is the average lifetime of the chaotic transients caused by the chaotic saddle. Since trajectories escape from the chaotic saddle along the unstable manifold, at large positive time n, *the $N(n)$ trajectory points will be in the vicinity of the unstable manifold. Let* C be a small box within S that contains part of the unstable manifold. The natural measure associated with the unstable manifold in C can thus be defined as [19,23]

$$\mu_u(C)=\lim_{n\to+\infty}\lim_{N_0\to\infty}\frac{N_u(n,C)}{N(n)}, \tag{2}$$

where $N_u(n,C)$ is the number of $N(n)$ orbits in C at time n. Similarly, the natural measure of the stable manifold in a box C in S can be defined as [19,23]

$$\mu_s(C)=\lim_{n\to+\infty}\lim_{N_0\to\infty}\frac{N_s(n,C)}{N(n)}, \tag{3}$$

where $N_s(n,C)$ is *the number of initial conditions* in C whose trajectories do not leave S before time n.

From the definitions Eqs. (2) and (3), we see that the natural measure associated with the stable and the unstable manifolds in C are determined by the number of trajectory points in C at time zero and time n, respectively. The natural measure of the chaotic saddle, μ, can then be defined by considering $N_m(\rho,n,C)$, the number of trajectory points in C at a time ρn in between zero and n:

$$\mu(C)=\lim_{n\to+\infty}\lim_{N_0\to\infty}\frac{N_m(\rho,n,C)}{N(n)}, \tag{4}$$

where $0<\rho<1$, $N_m(0,n,C)=N_s(n,C)$, and $N_m(1,n,C)=N_u(n,C)$. For large N_0 and n, trajectories that remain in S would stay near the chaotic saddle for most of the time between zero and n, except at the beginning when they are attracted towards the saddle along the stable manifold, and at the end when they are exiting along the unstable manifold. Thus, the measure defined in Eq. (4) is independent of ρ, as long as $0<\rho<1$.

Note that although $N(n)$ decreases exponentially in time, this decaying factor has been compensated in the definitions of the natural measures Eqs. (2)–(4). These measures are thus invariant under the dynamics, and they are called the *conditionally invariant measures* [3]. Numerically, the natural measure of the chaotic saddle can be computed by using the sprinkler method [23] or the proper-interior-maximum (PIM) -triple method [24], the latter usually generates long trajectories on the chaotic saddle. Dynamical invariants of the chaotic saddle, such as the fractal dimensions and the Lyapunov exponents, can then be defined with respect to the conditionally invariant measure of the saddle.

To estimate the contribution to the natural measure by unstable periodic orbits of period p, we consider a time $n>p$ at which we wish to examine how many trajectory points, out of those from N_0 initial conditions, still remain in C. For hyperbolic chaotic sets, at time p, the fraction of trajectory points that are still in B is given by [19] $\Sigma_{\mathbf{x}_{ip}\in C}1/L_1(\mathbf{x}_{ip})$, where the summation is over all unstable fixed points $\mathbf{x}_{ip}$ of the p-time iterated map contained in C and $L_1(\mathbf{x}_{ip})$ is the magnitude of the unstable eigenvalue of ith fixed point. Since $p<n$, we have

$$N_m(\rho,n,C)=N_0\sum_{\mathbf{x}_{ip}\in C}\frac{1}{L_1(\mathbf{x}_{ip})},$$

where $\rho=p/n<1$. The natural measure of the chaotic saddle contained in C is thus given by

$$\mu(C)=\lim_{n\to+\infty}\lim_{N_0\to\infty}\frac{N_m(\rho,n,C)}{N(n)}=\lim_{p\to\infty}\sum_{\mathbf{x}_{ip}\in C}\frac{\exp(p/\tau)}{L_1(\mathbf{x}_{ip})}. \tag{5}$$

Since S is a phase-space region that contains the whole chaotic saddle, we have $\mu(S)=1$, which, from Eq. (5), gives [25]

$$\mu_S(p)\equiv\lim_{p\to\infty}\sum_{\mathbf{x}_{ip}\in S}\frac{1}{L_1(\mathbf{x}_{ip})}=\exp(-p/\tau). \tag{6}$$

Equations (5) and (6) are rigorously valid only for hyperbolic chaotic saddles. The applicability of these equations to nonhyperbolic chaotic saddles remains, thus, only a conjecture. To provide numerical evidence, we choose the Hénon map [26]: $(x,y)\to(a-x^2+by,x)$, where a and b are parameters for which systematic computations of unstable periodic orbits can be done [27]. To obtain chaotic saddles, we fix $b=0.3$ and choose $a>a_c$, where $a_c\approx1.426$ is the crisis value beyond which the Hénon chaotic attractor is converted into a chaotic saddle [1]. For $a\gtrsim a_c$, explicit numerical computation reveals that the minimum possible angles between the stable and the unstable directions for points on the chaotic saddles can be arbitrarily close to zero, indicating that the chaotic saddles are nonhyperbolic [22]. Here we report our results with four values of a, the chaotic saddles at which are apparently nonhyperbolic [22]. We compute all unstable periodic orbits of periods up to 28 by using the algorithm in Ref. [27] and their eigenvalues. The quantity $\mu_S(p)$ in Eq. (6) is then computed as a function of p. Figure 1(a) shows $\ln\mu_S(p)$ versus p for $a=1.6$. We observe that $\mu_S(p)$ decays exponentially, as predicted by Eq. (6). The slope of the fit is $\kappa\equiv1/\tau=0.08\pm0.008$, which gives $\tau=12.5\pm1.3$. From a direct numerical realization of Eq. (1), we obtain $\tau\approx11.2\pm0.1$, which agrees well with that from periodic orbits, as in Fig. 1(a). Figure 1(b) shows the values of τ obtained via Eqs. (6) and (1) for $a=1.5$, 1.55, 1.6, and 1.65. The closeness of the values of τ obtained via these two independent approaches indicates the applicability of Eq. (6) for nonhyperbolic chaotic saddles. To check the validity of Eq. (5), we divide the phase-space region: $-2\leqslant(x,y)\leqslant2$, in which the chaotic saddles lie by a grid of 128×128. We use the PIM-triple algorithm to generate a long trajectory on the saddle [24], identify nonempty cells that the trajectory visits, and compute the frequency of visits, or the approximation to the natural measure, in each nonempty cell. Call this natural measure μ_i, $i=1,\ldots,N_{ne}$, where N_{ne} is the number of nonempty cells. The contribution to the natural measure, $\mu_i(p)$, from each nonempty cell by all periodic orbits of period p contained in the cell, is then computed. Figure 2(a) shows $\ln\Delta\mu(p)$ versus p for $a=1.6$, where $\Delta\mu(p)$

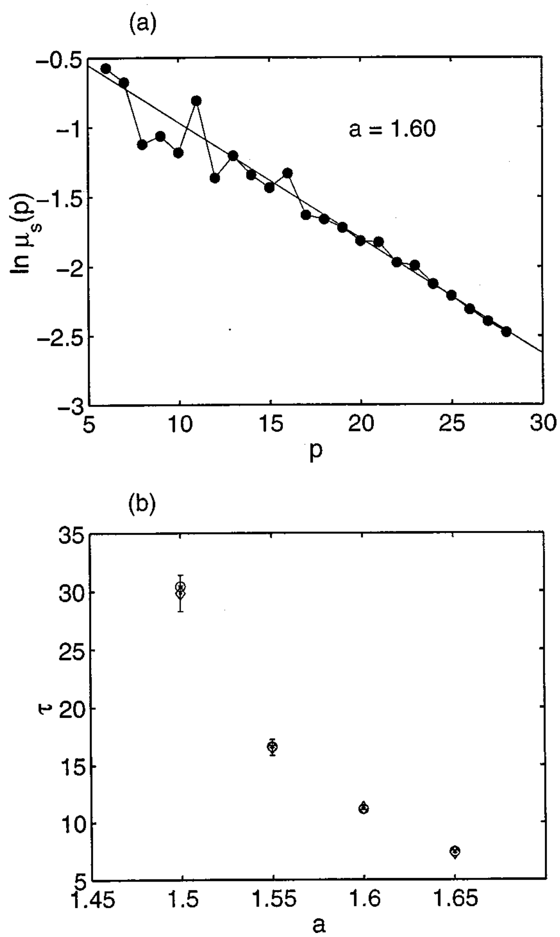

FIG. 1. (a) For the Hénon chaotic saddle at $a=1.6$ and $b=0.3$, $\ln \mu_S(p)$ vs p. (b) Comparison of the lifetimes of a chaotic saddle obtained by extracting the slopes of the lines of $\ln N(n)$ vs n (circles) and those obtained via Eq. (6) through unstable periodic orbits (diamonds) for $a=1.5$, 1.55, 1.6, and 1.65.

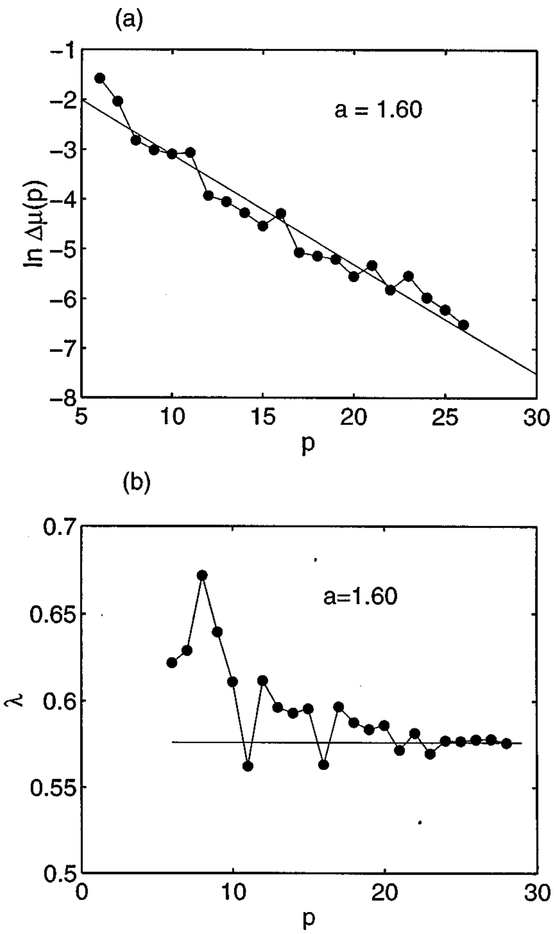

FIG. 2. (a) For the Hénon chaotic saddle at $a=1.6$, $\ln \Delta\mu(p)$ vs p. (b) The positive Lyapunov exponent estimated from a long PIM-triple trajectory (the horizontal line) and those from unstable periodic orbits (dots).

$\equiv(1/N_{ne})\Sigma_{i=1}^{N_{ne}}|\mu_i-\mu_i(p)|$. We see that $\Delta\mu(p)$ decreases exponentially as p increases, indicating the validity of Eq. (5) for large periods.

The main source of error in the exponential fittings in Figs. 1(a) and 2(a) comes from the fact that Eqs. (5) and (6) are theoretically valid only for hyperbolic chaotic saddles in the limit of $p\rightarrow\infty$. While we believe that Eqs. (5) and (6) are also valid for nonhyperbolic chaotic saddles, we are not aware of any theoretical tools that can be utilized to derive these equations when there are tangency points on the saddle between the stable and the unstable manifolds. Due to practical limitation, numerical verification of Eqs. (5) and (6) is possible only for periodic orbits of finite periods less than, say, 30. We are satisfied that numerical experiments performed with periodic orbits of periods up to about 28 already give robust fittings to the conjectured exponential behavior in Eq. (6). Theoretically, for hyperbolic chaotic saddles, it can be argued that the error in the exponential fitting decreases exponentially as the period increases [20]. Numerically, we find that other ways of fitting the data in Figs. 1(a) and 2(a), such as power-law fitting, are apparently ruled out [21].

Equation (5) implies that statistical averages of dynamical invariants and physical functions with respect to the natural measure of the chaotic saddle can be computed in terms of unstable periodic orbits embedded in the saddle. This offers an alternative way to check the validity of Eq. (5). To do this, we compute the positive Lyapunov exponent of the chaotic saddle by using (i) a long PIM-triple trajectory, and (ii) by using periodic orbits, as shown in Fig. 2(b), where the horizontal line denotes the exponent from the PIM-triple trajectory and dots are those from all periodic orbits of period p. Apparently, the exponent estimated from the periodic orbits asymptotes to the one from the PIM-triple trajectory, and we observe that the difference decreases exponentially as p increases. The results summarized in Figs. 1 and 2 thus strongly suggest the validity of the periodic-orbit characterization of the natural measure for nonhyperbolic chaotic saddles.

In summary, we provide strong evidence for the applicability of the periodic-orbit theory of the natural measure for an important class of dynamical invariant sets: nonhyperbolic chaotic saddles. As in the case of chaotic attractors, the natural measure is important because it is the one that is usually produced in physical experiments involving transient chaos. Our systematic numerical computations suggest that the characterization of this measure by unstable periodic orbits, while previously shown to be valid for hyperbolic saddles [19], is apparently correct for nonhyperbolic chaotic saddles in dissipative dynamical systems as well. The periodic-orbit theory is conceptually appealing and is potentially useful for further theoretical or even practical developments [28].

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