Targeted control of amplitude dynamics in coupled nonlinear oscillators

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We propose a general strategy for designing coupling functions in order to achieve a desired amplitude dynamics in coupled nonlinear oscillators. The target dynamics achieved by the proposed control schemes is a fixed-point motion at a desired amplitude level or a periodic motion at a desired frequency. The control schemes are illustrated with Rössler and Hindmarsh-Rose oscillators.

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Over the last two decades, control of dynamical systems and stabilization of unstable dynamical states have been a topic for intense research in theoretical and experimental nonlinear science [1]. Ott, Grebogi, and Yorke [2] introduced a control technique to stabilize a chaotic system in the neighborhood of a desirable unstable periodic orbit naturally embedded in the chaotic motion. Pyragas [3] proposed a time-delayed self-controlling feedback technique. Experimental control of chaos by one or both of these methods has been achieved in a variety of chaotic and stochastic systems, including turbulent fluids, oscillating chemical reactions, magnetomechanical oscillators, cardiac tissues, and neurons [4]. Here, for physically accessible or modifiable coupling functions, we propose a general strategy for designing coupling functions in order to achieve a desired amplitude dynamics in coupled chaotic systems.

The key observation is that an appropriate coupling function can put a coupled system of chaotic oscillators to a fixed-point motion and time-dependent sinusoidal inputs in the coupling functions can further stabilize the system to a targeted periodic orbit. The stabilization of fixed points in coupled oscillators is a phenomenon that has been commonly known as amplitude death (AD) [5–9]: as a consequence of the interaction, a pair of fixed points becomes stable and attracting, leading to a loss of oscillatory dynamics. Amplitude death was first observed experimentally in diffusively coupled chemical systems [5], and by now is known in a variety of other situations [8]. The stabilization of fixed points has been an important objective of many experimental studies. In coupled laser systems, for instance, the removal of chaotic fluctuations is highly desirable [10,11]. When the systems are identical, AD is known to occur when there is time delay in the coupling [7], a scenario that can occur even when the systems are not identical [6]. A series of studies of delay coupled systems have analyzed AD both theoretically and experimentally [8,9,11,12], and a novel setting of AD is via conjugate coupling where oscillators are coupled through dissimilar or conjugate variables [13].

Here, we propose a general strategy for achieving amplitude death and a purely periodic motion through the design of the coupling *function* in coupled chaotic systems. With this approach, AD can be observed in both identical and mismatched coupled oscillators, and with either instantaneous or time-delayed interactions. AD is thus generic in coupled systems, and by proper choice of coupling function essentially arbitrary specified fixed points can be stabilized. Results are presented for coupled Rössler oscillators as well as for model neurons with synaptic coupling. In a straightforward extension, specific periodic dynamics can also be stabilized, thus enabling control of complex dynamical systems.

It is widely recognized that natural systems are rarely isolated and that coupling interactions give rise to phenomena such as synchronization, hysteresis, phase locking, phase shifting, phase-flip, riddling or amplitude death [14,15]. Since nonlinear systems arise in a variety of contexts, such dynamics is of broad relevance a variety of areas of research in the physical, biological, and social sciences. Existing scenarios of amplitude death pertain to the stabilization of existing fixed points that are unstable in the uncoupled systems. Our approach, outlined below, offers a method for stabilizing *arbitrary* fixed points or periodic orbits.

Consider the case of two coupled oscillators

$$d\mathbf{X}_i/dt = \mathbf{F}_i(\mathbf{X}_i) + \epsilon \mathbf{G}(\mathbf{X}), \quad i = 1,2, \tag{1}$$

where $\mathbf{X}_i$ denotes the set of dynamical variables of the ith oscillator, and $\mathbf{G}(\mathbf{X}) = \mathbf{G}(\mathbf{X}_i, \mathbf{X}_j)$ is the coupling function between ith and jth oscillators. If the dynamics is oscillatory for $\epsilon = 0$, then the situation of amplitude death occurs when the coupling is sufficient to cause a stationary solution of the above system, say $\mathbf{X}_i^{(0)}$, to become stable and attracting. When the coupling $\mathbf{G}$ is diffusive, these fixed points are typically also stationary solutions of the uncoupled system, namely, they are determined by the condition,

$$\mathbf{F}_i(\mathbf{X}_i^{(0)}) = 0, \tag{2}$$

since in this case the effective coupling also vanishes,

$$\mathbf{G}(\mathbf{X}^{(0)}) = 0. \tag{3}$$

The above scenario for AD has been studied extensively for a variety of situations: when the systems are identical [13], when the parameters of the two oscillators are not identical [6], when the linear coupling function $\mathbf{G}$ includes time delay [7,12], and so on. The basic conditions under which AD ensues are well understood [8].

The case when the coupling function **G** is nonlinear has been less extensively studied [16], although in a variety of natural systems [10,11], such coupling is unavoidable. While AD can still occur—and is a desired goal of some studies [11]—the fixed points that are stabilized no longer correspond to the stationary points of the uncoupled system [12,13].

Here we find that new desired fixed point that can be created and stabilized through an appropriate coupling function. The essence of the procedure is the following. Given a set of desired fixed points, $\bar{\mathbf{X}}_i$, these will be stationary points of the coupled system with an additional constant source, namely, of the modified dynamical system

$$d\mathbf{X}_i/dt = \mathbf{F}_i(\mathbf{X}_i) + \epsilon\mathbf{G}(\mathbf{X}) - \mathbf{F}_i(\bar{\mathbf{X}}_i). \quad (4)$$

The source function $\mathbf{F}_i(\bar{\mathbf{X}}_i)$ takes a constant value that depends on the desired fixed points. For suitable **G** it can be arranged that $G(\bar{\mathbf{X}})=0$. Upon variation of the coupling parameters such as the coupling strength ϵ (or by including time-delay τ in **G**), the new fixed point can be stabilized: this, effectively, is *targeted* amplitude death.

As an illustration we consider coupling between two identical chaotic Rössler oscillators [17] through an exponential term, $\mathbf{G}\equiv[(x_i-\beta)\exp(x_j-\delta),0,0]^T$ [16] where T denotes the transpose. The resulting equations for the coupled system are $(i,j=1,2,i\neq j)$

$$\dot{x}_i = -y_i - z_i - \epsilon(x_i-\beta)\exp(x_j-\delta),$$

$$\dot{y}_i = x_i + ay_i,$$

$$\dot{z}_i = b + z_i(x_i - c). \quad (5)$$

Here the coupling is via the variables x_1 and x_2, and we have introduced the parameters β and δ in **G**. Clearly, $\mathbf{G}=\mathbf{0}$ for $x_i=\beta$ and examination of the dynamical equations suggests that a fixed point $\bar{x}_i=\beta$, $\bar{y}_i=-\beta/a$ and $\bar{z}_i=-b/(\beta-c)$ can be created by modifying the above equations to

$$\dot{x}_i = -y_i - z_i - \epsilon(x_i-\beta)\exp(x_j-\delta) + (\bar{y}_i+\bar{z}_i),$$

$$\dot{y}_i = x_i + ay_i,$$

$$\dot{z}_i = b + z_i(x_i - c). \quad (6)$$

The stability of this fixed point can be examined [18] as a function of ϵ and β. In Fig. 1 the stable (S) and unstable (U) regions are indicated: the unstable solution corresponds to the unbounded motion of the system while stable regime indicates the possibility of AD solution. Fixing $\beta=1$, we compute the largest real part of eigenvalue, Re(λ) [18] at the fixed point; this is shown in Fig. 2(a) and it is clear that amplitude death can occur when Re(λ) becomes negative. Similarly, shown in Fig. 2(b) are transients for different values of β with $\epsilon=0.05$. In all cases, the desired fixed point $\bar{x}_i=\beta$ is achieved.

It is a simple extension of this idea to stabilize arbitrary $\bar{x}_i$, $\bar{y}_i$, $\bar{z}_i$ via the modified dynamical equations

$$\dot{x}_i = -y_i - z_i - \epsilon(x_i-\beta)\exp(x_j-\delta) + (\bar{y}_i+\bar{z}_i),$$

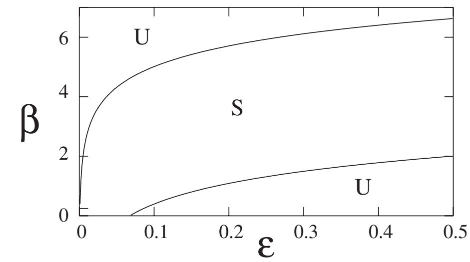

FIG. 1. Schematic phase diagram in the ϵ-β plane for the coupled Rössler system, Eq. (6).

$$\dot{y}_i = x_i + ay_i - (\bar{x}_i + a\bar{y}_i),$$

$$\dot{z}_i = b + z_i(x_i - c) - \{b + \bar{z}_i(\bar{x}_i - c)\}, \quad (7)$$

and examining the dynamics as a function of the remaining parameters, ϵ and δ. An example of such a targeted amplitude death is shown in Fig. 3 for different desired fixed points.

Exponential coupling [as taken in Eq. (5), for example] arises naturally in neuroscience where it has been extensively studied in the context of synaptic coupling. Consider Hindmarsh-Rose (HR) [19] neurons,

$$\dot{x}_i = d_1x_i^2 - x_i^3 - y_i - z_i - \epsilon g(x_i,x_j) - f(\bar{\mathbf{X}}_i),$$

$$\dot{y}_i = (d_1 + d_2)x_i^2 - y_i,$$

$$\dot{z}_i = d_3(d_4x_i + d_5 - z_i), \quad (8)$$

with

$$g(x_i,x_j) = \frac{(x_i-\beta)}{1+\exp\{-\gamma(x_j-\Theta_s)\}}, \quad (9)$$

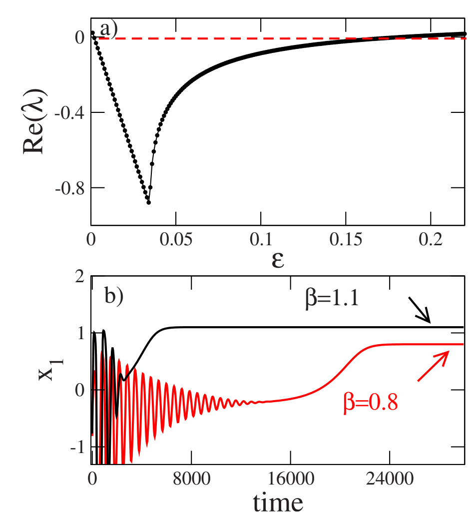

FIG. 2. (Color online) (a) Real part of the largest eigenvalue as a function of the coupling strength ϵ at $\beta=1$. (b) Transient trajectories, x_1 vs time for $\beta=0.8$ (dashed line) and 1.1 (solid line) at coupling strength $\epsilon=0.05$.

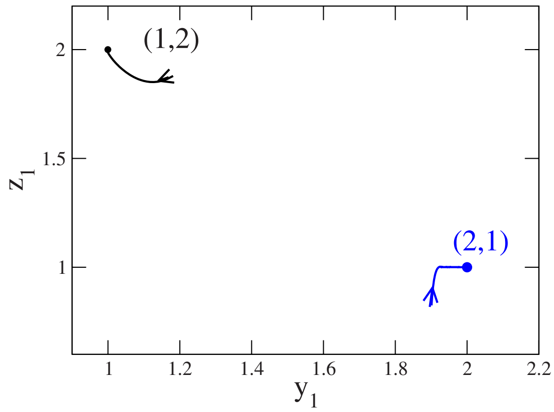

FIG. 3. (Color online) Transient trajectories in the y-z plane leading to the specified values of $\bar{y}_1$ and $\bar{z}_1$ at $\bar{x}_1=\beta=1$ and $\epsilon=0.05$ for the coupled Rössler system, Eq. (7).

$$f(\bar{\mathbf{X}}_i) = d_1\bar{x}_i^2 - \bar{x}_i^3 - \bar{y}_i - \bar{z}_i. \tag{10}$$

where $\mathbf{G}(\mathbf{X})=[g(x_i,x_j),0,0]^T$. Here the x's are membrane potentials while the variables y and z are associated with fast and slow currents, respectively. The external parameter is the strength ϵ of the sigmoidal synaptic coupling. The reversal potential, β and the spiking threshold is held fixed. Θ is a synaptic threshold and γ controls the slope in the exponential function. The fixed points of the coupled system are $\bar{x}_i=\beta$, $\bar{y}_i=(d_1+d_2)\beta^2$, $\bar{z}_i=d_2\beta+d_5$. This fixed point differs from those of the uncoupled system. Eigenvalues [18] of the stability matrix at this fixed point reveals that, as shown in Fig. 4, there is a small region where oscillatory motion (OS) persist for small coupling strength while AD occurs in the remaining region. The details of largest eigenvalue of the fixed point is plotted in Fig. 5(a) which clearly shows that for small value of coupling strength there oscillating motion [Re($\lambda>0$) while at higher values there is existence of AD (having negative values)]. The transient trajectories for controlled fixed point $\beta=1$ and 2 are shown in Fig. 5(b) at coupling strength $\epsilon=2$. Note that in both systems, Eqs. (6) and (8), AD occurs even though the subsystems are identical and instantaneously coupled [16].

Here we also show that it is possible to target periodic orbits by this method. Modification of the coupling function $G\rightarrow\{x_i-\beta\sin(\omega t)\}$ gives a targeted periodic orbit of frequency ω in a specific range of parameters. Shown in Figs. 6(a) and 6(b) are the variation of resulting common frequency Ω of the two oscillators for a given forcing frequency ω in a coupled Rössler system, Eq. (6) and HR system, Eq. (8), respectively. Here, the parameters are fixed such that in the absence of periodic forcing, amplitude death will result. The input and output frequencies are identical, showing that one can indeed target periodic motion as desired. The inset figures in Fig. 6 shows the time series (x_1 vs time) associated with such a target periodic motions for $\omega=5$.

In summary, the main result presented here is that, with specific coupling, the stabilization of the new fixed points can occur in the absence of time delay even when the interacting systems are identical. Recognition of the manner in which coupling causes amplitude death leads to a strategy for the stabilization of *arbitrary* fixed points by a suitable modification of the coupling function [21]. A straightforward extension of the procedure is effective in stabilizing networks:

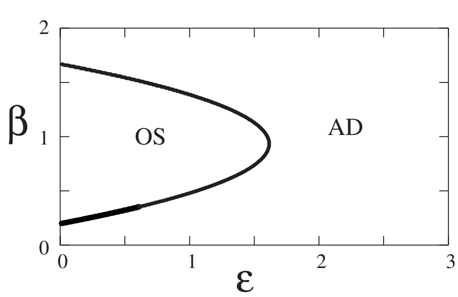

FIG. 4. Schematic phase diagram in the ϵ-β plane for the coupled HR system, Eq. (8). The other parameters are fixed at $d_1=2.8$, $d_2=1.6$, $d_3=0.001$, $d_4=9$, $d_5=5$, $\Theta=-0.25$, and $\gamma=10$ [20].

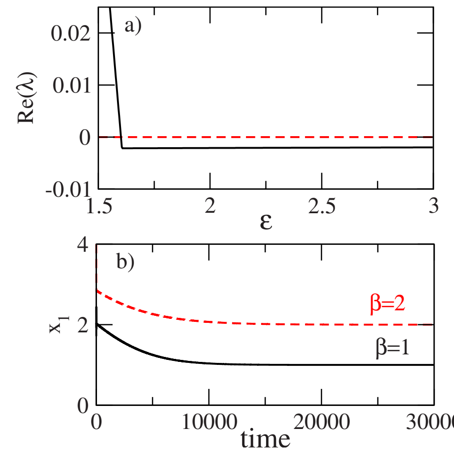

FIG. 5. (Color online) (a) Real part of the largest eigenvalue as a function of the coupling strength ϵ at $\beta=1$ for HR system, Eq. (8). (b) Transient trajectories, x_1 vs time for $\beta=2$ (dashed line) and 1 (solid line) at coupling strength $\epsilon=2$.

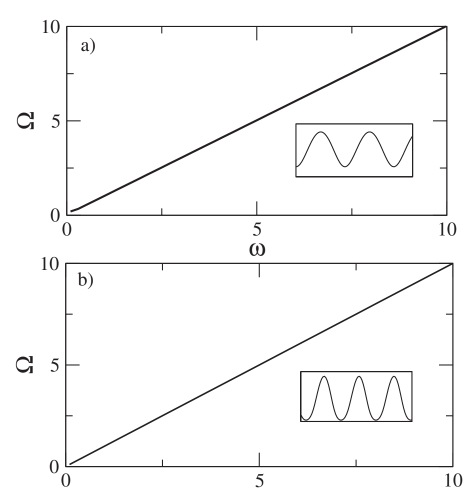

FIG. 6. The variation of the output frequency of coupled (a) Rössler ($\beta=0.4$ and $\epsilon=0.01$) and (b) HR ($\beta=1$ and $\epsilon=2$) oscillators forced harmonically with targeted frequency ω.

we have also verified this for sets of 10 coupled oscillators (both Rössler and HR) with nearest-neighbor interactions and periodic boundary conditions (figures not included here). We have also observed similar phenomena with finite delay and mismatched oscillators. Furthermore, this strategy is effective in creating oscillatory periodic motion of desired frequency. The control schemes presented here can be of considerable interest in controlling undesirable dynamic fluctuations in the output of a coupled physical or engineering system. A specific example is the case of semiconductor lasers where different coupling schemes have been proposed to stabilize low-frequency chaotic fluctuations [11]. Similarly, creating a periodic oscillation at a specific frequency can be of tremendous interest for a variety of physical and biomedical applications.

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