7 January 2002

PHYSICS LETTERS A

ELSEVIER

Physics Letters A 292 (2002) 269–274

www.elsevier.com/locate/pla

Generalized stability law for Josephson series arrays

Mukeshwar Dhamala, Kurt Wiesenfeld *

School of Physics, Georgia Institute of Technology, Atlanta, GA 30332, USA

Received 30 October 2001; accepted 7 November 2001
Communicated by C.R. Doering

Abstract

By deriving an N-dimensional Poincare map, we explore the enhanced stability of Josephson series arrays using capacitive junctions. The analytic expression for the critical Floquet multiplier has a direct physical interpretation, affording new insight into the conditions that affect inphase stability. In particular, we generalize the well-known stability principle previously established for arrays of zero capacitance junctions. Optimally capacitive junction arrays offer a much improved candidate for experimental realization of the Kuramoto model. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

The study of Josephson junction arrays is almost as old as the Josephson effect itself [1–3]. Interest in arrays has remained high due to a number of potential applications as detectors, oscillators and amplifiers in very high frequency electronics [4,5]. For example, Josephson arrays are currently used in high sensitivity magnetic flux detectors in biology and biomedicine [6], and finding new applications in high sensitivity scanning microscope to monitor aircraft corrosion [7], in tunable local oscillators at millimeter and submillimeter wavelengths [8] and in parametric amplifiers [9]. They are also presently used to maintain the U.S. Legal Volt [10]. Since fast digital logic circuits can be constructed from Josephson junctions, there are other possibilities of new applications in the areas of computers, telecommunications and remote sensing. In the meantime, Josephson arrays have become a popular example in nonlinear dynamics, serving as an archetype of nonlinear coupled oscillators and the various ordered and chaotic phenomena exhibited by such systems.

These two motivations intersect in the phenomenon of spontaneous synchronization. Here, an array is driven by a constant bias current which induces voltage oscillations in each junction, due to the nonlinear nature of the Josephson effect. The simplest collective behavior is also the most desirable (in most applications), wherein all junctions oscillate with the same frequency and in phase. By far, theoretical progress has been greatest for series arrays of zero-capacitance junctions, a class of problems which has proved to be surprisingly tractable. Such arrays can be mapped onto the Kuramoto model familiar from dynamical systems theory [11]; they also possess a remarkable dynamical structure typically associated with Hamiltonian systems despite their manifestly dissipative nature [12]. There is also a crisp physical condition for when inphase oscillations are stable, hinging simply on whether the coupling load looks inductive at the operating frequency [13–15].

* Corresponding author.
E-mail address: kurt.wiesenfeld@physics.gatech.edu (K. Wiesenfeld).

0375-9601/02/$ – see front matter
PII: S0375-9601(01)00790-3

Our understanding of spontaneous synchronization of capacitive junctions is far less advanced. Although the best power levels have been achieved for junctions with zero capacitance [16,17], the potential advantage of using junctions with nonzero capacitance is that they may perform better, simply because there is another parameter to vary. Indeed, numerical studies by Hadley and co-workers [18], predicted that inphase stability is greatest for junctions with McCumber parameters of about unity. This observation was observed to be roughly independent of the load properties, and no physical explanation has ever been offered for it. Nor does the physical condition for inphase stability carry over from the zero-capacitance junction case, and sometimes dramatically: an array of capacitive junctions with a purely capacitive load admits a range of inphase stability, even though the load is (obviously) capacitive at all frequencies.

In this Letter, we are able to address all of these issues. Extending the work of Chernikov and Schmidt [19], we derive an N-dimensional return map, from which we calculate the Floquet multipliers of the inphase solution. The multipliers provide a quantitative measure of stability and thus allow us to find the conditions corresponding to maximum stability. The resulting expression admits a direct physical interpretation, providing fresh insight into the nature of spontaneous synchronization in these arrays. We also show that the optimally stable array is a significantly better candidate for experimental realization of the Kuramoto model.

We first consider a purely capacitive load since the analysis is relatively compact, and also best highlights the role of junction capacitance in rendering the inphase state stable. Consider a series array of N identical junctions, biased with a constant current and shunted by a capacitor. The circuit equations can be put into the following dimensionless form:

$$\beta\ddot{\phi}_k + \dot{\phi}_k + b\sin\phi_k + \alpha\sum_{j=1}^{N}\ddot{\phi}_j = 1, \tag{1}$$

where ϕ_k is the wave function phase difference across the kth junction, β is proportional to the junction capacitance, b^{-1} is the bias current, and αN is the load capacitance. Following Chernikov and Schmidt, we expand in powers of b, letting $\phi_k = \phi_k^{(0)} + \phi_k^{(1)} + \phi_k^{(2)} + \cdots$, where $\phi_k^{(n)} \sim b^n$, thereby generating a sequence of differential equations for successive powers of b. These admit the following solutions for the $\phi_k^{(n)}$:

$$\phi_k^{(0)} = t + \theta_k, \tag{2a}$$

$$\phi_k^{(1)} = A_k\sin t + B_k\cos t, \tag{2b}$$

$$\phi_k^{(2)} = \frac{1}{2}b(A_k\sin\theta_k - B_k\cos\theta_k)t + C_k\sin 2t + D_k\cos 2t, \tag{2c}$$

where θ_k is a constant of order unity, A_k, B_k, C_k, D_k are constants of order b. Since $\phi_k^{(n)} \sim b^n$, the higher the bias current, b^{-1}, the better the convergence of the solutions. Determining explicit expressions for these constants in terms of the θ_k is straightforward. For example, one finds

$$\frac{1}{2}b(A_k\sin\theta_k - B_k\cos\theta_k) = \omega - \frac{\alpha\omega}{K}\left[r_1\sum_j\sin(\theta_j - \theta_k) + r_2\sum_j\cos(\theta_j - \theta_k)\right], \tag{3}$$

where $\omega = b^2/2(1+\beta^2)$, $K = (\beta + \alpha N)^2 + (1 + \alpha N)^2$, $r_1 = 1 - \beta^2 - \beta\alpha N$, and $r_2 = 1 - \beta^2 + 2\beta + \alpha N$.

At this point we diverge from Ref. [19] and explicitly deduce an N-variable return map. From Eqs. (2), we evaluate ϕ_k at times $t = 0$ and 2π:

$$\phi_k(0) = \theta_k + B_k + D_k, \tag{4}$$

$$\phi_k(2\pi) = 2\pi + \theta_k + B_k + \frac{1}{2}b(A_k\sin\theta_k - B_k\cos\theta_k)2\pi + D_k. \tag{5}$$

Using Eq. (3) and noting that $\phi_j(0) = \theta_j + O(b)$, we arrive at a map correct through $O(b^2)$:

$$\phi_k \to \phi_k + \Omega - \frac{K_1}{N}\sum_j\sin(\phi_j - \phi_k) + \frac{K_2}{N}\sum_j\cos(\phi_j - \phi_k), \tag{6}$$

where $\Omega = 2\pi + \omega$, $K_1 = 2\pi\alpha N\omega r_1/K$, and $K_2 = 2\pi\alpha N\omega r_2/K$. For the inphase solution, $\phi_j - \phi_k = 0$, so that $\phi_k = n(\Omega + K_2)$ for all k and $n = 1, 2, 3, \ldots$. To test its stability, we apply the perturbations ϵ_k and

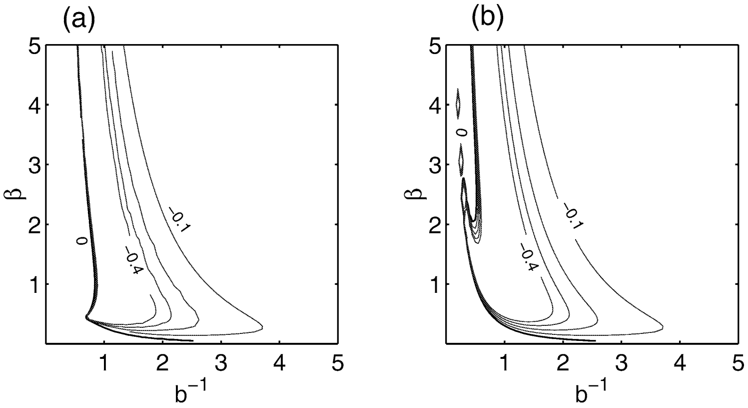

Fig. 1. Comparison of (a) numerical computation and (b) analytical predictions: contours of the largest Floquet exponent are plotted as a function of the junction capacitance β and the bias current b^{-1} for a purely capacitive load $C = 3$. Notice that there is excellent agreement between (a) and (b) at higher bias current b^{-1}: the difference between the the values of the exponents in (a) and (b) reduces from 5.5% to 0.35% when b^{-1} is changed from 1.4 to 2.8.

examine the linearized map for their evolution,

$$\epsilon_k' = (1 + K_1)\epsilon_k - \frac{K_1}{N}\sum_j \epsilon_j. \tag{7}$$

The change of variables $\delta_k = \epsilon_{k+1} - \epsilon_k$ and $H = \sum_k \epsilon_k$ diagonalizes this map, and the Floquet multipliers can be read off. One multiplier equals to one and the other $(N - 1)$ are equals to $1 + K_1$, or expressed in the original system parameters:

$$\mu = 1 + \frac{\pi\alpha N b^2}{1+\beta^2}\frac{1-\beta^2-\beta\alpha N}{(\beta+\alpha N)^2+(1+\alpha N)^2}. \tag{8}$$

Since αN is the load capacitance, μ is independent of N. The inphase orbit is stable if μ has magnitude less than one; the smaller its magnitude the greater the stability. The associate Floquet exponent is $\rho = (\ln\mu)(\Omega/2\pi)$; physically, it gives the relaxation rate. Fig. 1 shows contour plots of the largest (nonzero) Floquet exponent as a function of the junction capacitance β and the bias current b^{-1}, comparing the derived formula against the value determined from Eq. (1) via direct numerical analysis. The agreement is quite good. Surprisingly, agreement remains reasonably good even for b^{-1} near unity, a piece of good fortune remarked on by Chernikov and Schmidt in their work on the stability boundary.

The contours corresponding to greatest stability occur for $\beta \approx 1$, in agreement with earlier numerical work [18,20]. The nearly vertical orientation of the contour means that maximum stability is, for all practical purposes, achieved over a range of β values. This is a welcome feature since it is difficult in practice to accurately "dial in" the junction capacitance.

In their work on anisotropic ladder arrays [21], Trees and Hussain drew a useful correspondence between optimized stability as a function of β and critical damping in linear oscillators. The analogy is not complete in our problem since although the transient response damps out most rapidly, there is no transition between monotonic and oscillatory decay of transients.

Result (8) is complicated, but we can get substantial insight by comparing the array problem with a single junction driven by a periodic bias current:

$$\beta\ddot{\phi} + \dot{\phi} + b\sin\phi = 1 + f\sin(t+\xi), \tag{9}$$

where b and f are small and of the same order. Repeating the steps of the previous perturbation calculation we generate, order by order, a sequence of differential equations. We can write the corresponding solutions this way:

$$\phi^{(0)} = t + \theta, \tag{10a}$$

$$\phi^{(1)} = -b\tilde{C}\sin(t+\theta+\zeta) + f\tilde{C}\sin(t+\xi+\zeta), \tag{10b}$$

$$\phi^{(2)} = \frac{1}{2}\big[b^2\tilde{C}\sin\zeta - bf\tilde{C}\sin(\xi-\theta+\zeta)\big]t$$

$$+\tilde{D}\sin(2t+\psi), \tag{10c}$$

where the quantities $\tilde{C}$ and ζ which characterize the response for a *unit amplitude* sinusoidal forcing, i.e., $\tilde{C}\sin(t+\zeta)$, where $\tilde{C}=1/\sqrt{1+\beta^2}$ and $\zeta=\arctan(1/\beta)$, and $\tilde{D}$ and ψ are constants whose explicit forms are not needed here. The return map is

$$\phi\to\phi+2\pi+\pi b^2\tilde{C}^2+\pi f b\tilde{C}\sin(\xi+\zeta-\phi). \tag{11}$$

An entrained solution satisfies $\phi^*\to\phi^*+2\pi$. There is one such stable solution and the associated Floquet multiplier is easily determined:

$$\mu=1-\pi f b\tilde{C}\cos(\xi+\zeta-\phi^*). \tag{12}$$

We see that the degree of stability depends on the product of the drive amplitude f and the response amplitude $\tilde{C}$, and on two other quantities: (1) $\xi-\phi^*$, which is the external drive phase relative to the overturning phase, and (2) the response phase shift ζ.

Now compare the array problem. The key step is to identify what corresponds to the ac-drive term of the single junction problem. This is readily done by considering the $O(b)$ differential equation for the array problem, which is

$$\beta\ddot{\phi}_j^{(1)}+\dot{\phi}_j^{(1)}=-b\sin\phi_j^{(0)}-\alpha\sum_k\ddot{\phi}_k^{(1)}. \tag{13}$$

The last term is just the current through the load, which we denote by J. Rather than being an externally imposed oscillation, J is self-consistently generated as the load response to the activated junctions. Multiplying Eqs. (13) through by α, summing over all j and differentiating twice yields

$$(\beta+\alpha N)\ddot{J}+\dot{J}=\alpha b\sum_k\sin(t+\theta_k). \tag{14}$$

Note that this depends on both the junction and load capacitances, β and α, and so describes the load response in situ, and not its response disembodied from the junction array. As before, we can write the solution for J in terms of the response if instead the right-hand side was a unit amplitude oscillation $\sin(t+\theta)$. Denoting the amplitude of response by $\tilde{D}$ and the phase shift of response by γ, we have $\tilde{D}=1/\sqrt{1+(\beta+\alpha N)^2}$ and $\sin\gamma=1/\tilde{D}$, so that

$$J=\alpha b\tilde{D}\sum_k\sin(t+\theta_k+\gamma). \tag{15}$$

Equating this to the driving term $f\sin(t+\xi)$ from the single junction problem, we have in the inphase state $f=\alpha b\tilde{D}N$ and $\xi=\theta+\gamma$. We can immediately write down the expression for the Floquet multiplier using Eqs. (12):

$$\mu=1-\pi b^2N\alpha\tilde{C}\tilde{D}\cos(\gamma+\zeta). \tag{16}$$

This result is quite interesting: the overturning phase has completely dropped out! The reason is that, in the array, the junctions are driven by the load oscillations, but those in turn are generated by the junction oscillations. Thus, the drive *relative* to the overturning phase is just $\xi-\theta=\gamma$.

Suppose the load is not purely capacitive. As written, result (16) is unchanged for a general *RLC* load [22], where now

$$\tilde{D}^{-2}=\left[(1-\mu_1)^2+\mu_2^2\right]\times\left[\left(\beta+\frac{\alpha N(1-\mu_1)}{(1-\mu_1)^2+\mu_2^2}\right)^2+\left(1+\frac{\alpha N\mu_2}{(1-\mu_1)^2+\mu_2^2}\right)^2\right], \tag{17}$$

where μ_1 is the dimensionless load inductance, μ_2 the resistance, and again $\sin\gamma=1/\tilde{D}$. The pure capacitive load is properly recovered when $\mu_1=\mu_2=0$. Eq. (16) gives a unified law for inphase stability,

$$\cos(\gamma+\zeta)>0,$$

which generalizes the well-known result in the $\beta=0$ limit. We recover that narrower result by noting that if $\beta=0$, the phase-shift $\zeta=\pi/2$, and the inphase state when $\sin\gamma<0$, which is equivalent to the condition that the system oscillation frequency is higher than the load's *LC* frequency.

Our generalized rule is almost as simple, but involves the physical properties of both the junction and the load, through their intrinsic phase shifts.

Finally, we return to the point that the N-dimensional return map (6) is valid for arbitrary sets of phases. If we allow small ($O(b^2)$) disorder in the junction parameters, the map is equivalent to the Kuramoto model [23]. This generalizes another $\beta=0$ result first noted a few years ago [11], that the Josephson series array is a physical realization of the Kuramoto system. This raised the possibility that the two order–disorder transitions of the Kuramoto model could be seen for

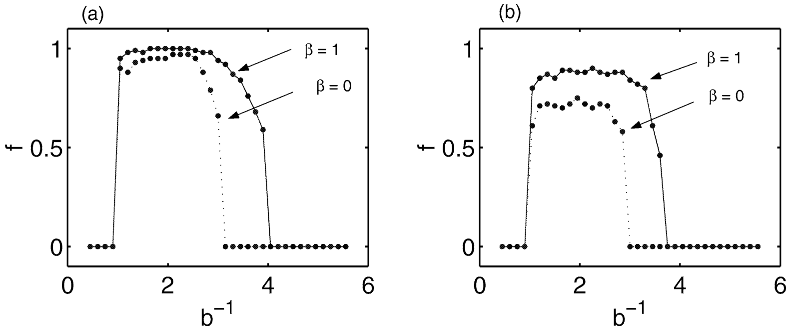

Fig. 2. Numerical simulations of disordered arrays using junctions with $\beta = 0$ and $\beta = 1$, showing fraction (f) of frequency-locked oscillators versus average bias current b^{-1} for $N = 100$, $L = 1.0$, $C = 3.0$ and $R = 1.0$ at (a) 0.1% and (b) 5.0% of disorder level in b^{-1}. (The disorder in b^{-1} is generated by a normally distributed random variable with mean at b^{-1} and variances of 0.1% and 5.0% for (a) and (b), respectively.)

the first time in a laboratory setting. The theory indicated that the second transition (onset of complete synchronization) was attainable using existing fabrication technology, though barely so. Our results broaden substantially the possibility of observing the Kuramoto transitions. With capacitive junctions, there are substantial regions in parameter space where the Floquet exponent is less than -0.4 with values reaching -1.5 in spots, compared with a best value of about -0.15 for zero-capacitance junctions (see Fig. 1). Since the exponent is directly proportional to the Kuramoto coupling constant [11], the transition to complete synchronization should occur for significantly higher levels of disorder. The numerical simulations shown in Fig. 2 underscore this point, with the transitions shifted by some factor and a higher number of locked junctions with $\beta = 1$ at the same level of disorder. This puts the required tolerances well within presently achievable limits.

Acknowledgement

This work was sponsored by the Office of Naval Research under contract No. N00014-99-1-0592.

References

[1] B.D. Josephson, Phys. Lett. 1 (1962) 251.
[2] T.D. Clark, Phys. Lett. A 27 (1968) 585.
[3] D.R. Tilley, Phys. Lett. 33A (1970) 205.
[4] K.A. Delin, T.P. Orlando, Foundations of Applied Superconductivity, Addison-Wesley, 1991.
[5] T. van Duzer, C.W. Turner, Principles of Superconductive Devices and Circuits, Elsevier, New York, 1981.
[6] J.P. Wikswo, IEEE Trans. Appl. Supercond. 5 (1995) 74; L.N. Vu, D.J. van Harlingen, IEEE Trans. Appl. Supercond. 3 (1993) 1918; J.R. Kirtley, M.B. Ketchen, C.C. Tsuei, J.Z. Sun, W.J. Gallagher et al., IBM J. Res. Dev. 39 (1995) 655; F.C. Wellstood, Y. Gim, A. Amar, R.C. Black, A. Mathai, IEEE Trans. Appl. Supercond. 7 (1996) 3134; W.G. Jenks, S.S.H. Sadeghi, J.P. Wikswo, Physica D 30 (1997) 293.
[7] J.R. Kirtley, J.P. Wikswo, Ann. Rev. Mater. Sci. 29 (1999) 117.
[8] S. Han, B. Baokang, W. Zhang, J.E. Lukens, Appl. Phys. Lett. 64 (1994) 1424; P.A.A. Booi, S.P. Benz, Appl. Phys. Lett. 68 (1996) 3799.
[9] E. Terzioglu, M.R. Beasley, IEEE Trans. Appl. Supercond. 5 (1995) 3349; B. Yurke, M.L. Roukes, R. Movshovich, A.N. Pargellis, Appl. Phys. Lett. 69 (1996) 3078.
[10] C.A. Hamilton, C. Burroughs, K. Chieh, J. Res. Natl. Inst. Stand. Technol. 95 (1990) 219.
[11] K. Wiesenfeld, P. Colet, S.H. Strogatz, Phys. Rev. Lett. 76 (1996) 404; K. Wiesenfeld, P. Colet, S.H. Strogatz, Phys. Rev. E 57 (1998) 1563.
[12] S. Watanabe, S.H. Strogatz, Physica D 74 (1994) 197.
[13] A.K. Jain, K.K. Likharev, J.E. Lukens, J.E. Sauvageau, Phys. Rep. 109 (1984) 309.
[14] P. Hadley, M.R. Beasley, Appl. Phys. Lett. 50 (1987) 621.
[15] K. Wiesenfeld, J.W. Swift, Phys. Rev. E 51 (1995) 1020.
[16] S. Han, B. Bi, W. Zhang, J.E. Lukens, Appl. Phys. Lett. 64 (1994) 1424.
[17] P.A.A. Booi, S.P. Benz, Appl. Phys. Lett. 68 (1996) 3799.
[18] P. Hadley, Ph.D. thesis, Stanford University (1989); P. Hadley, M.R. Beasley, K. Wiesenfeld, Phys. Rev. B 38 (1988) 8712.

[19] A.A. Chernikov, G. Schmidt, Phys. Rev. E 52 (1995) 3415.
[20] G.S. Lee, S.E. Schwarz, J. Appl. Phys. 55 (1984) 1035.
[21] B.R. Trees, N. Hussain, Phys. Rev. E 61 (2000) 6415.
[22] M. Dhamala, K. Wiesenfeld, unpublished.
[23] Y. Kuramoto, Lect. Notes Phys. 39 (1975) 420;
H. Sakaguchi, Y. Kuramoto, Prog. Theor. Phys. 76 (1986) 576.