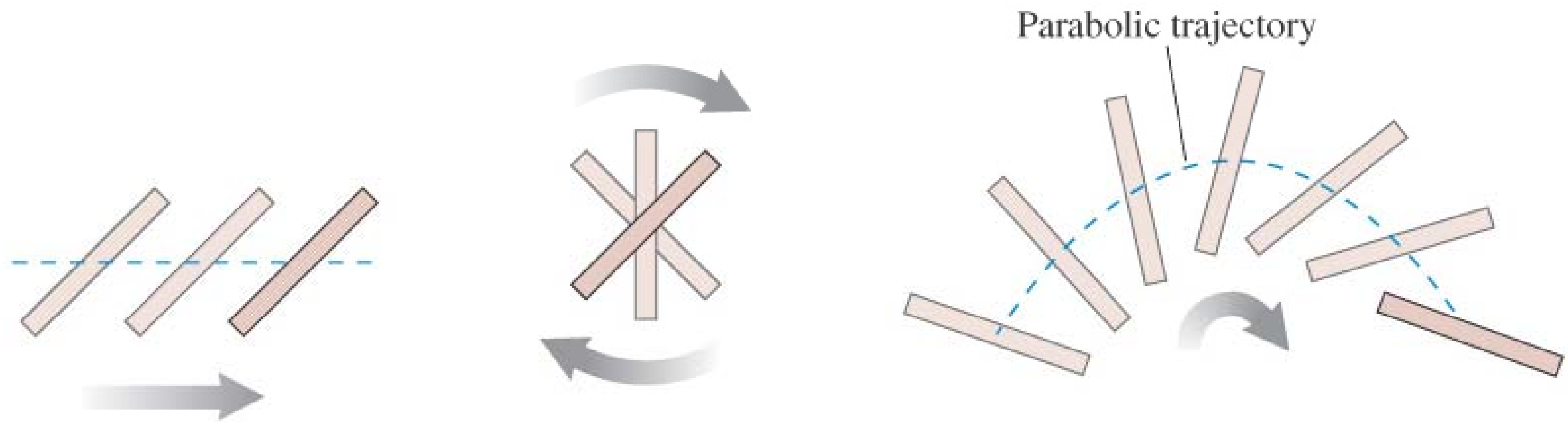
Rotation of a Rigid Body

Readings: Chapter 13



Translational motion: The object as a whole moves along a trajectory but does not rotate.

Rotational motion: The object rotates about a fixed point. Every point on the object moves in a circle.

Combination motion: An object rotates as it moves along a trajectory.

Newton's second law:

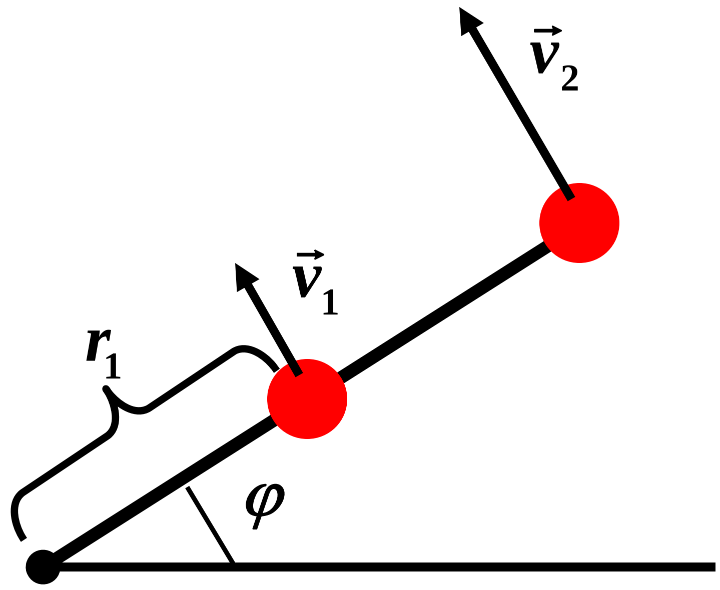
$$\vec{F} = m\vec{a}$$

How can we characterize the acceleration during rotation?

- translational acceleration and

- angular acceleration

Angular acceleration



Center of rotation

Both points have the same angular velocity $\omega = \dfrac{\Delta\varphi}{\Delta t}$

$$v_1 = \omega r_1 \qquad v_2 = \omega r_2$$

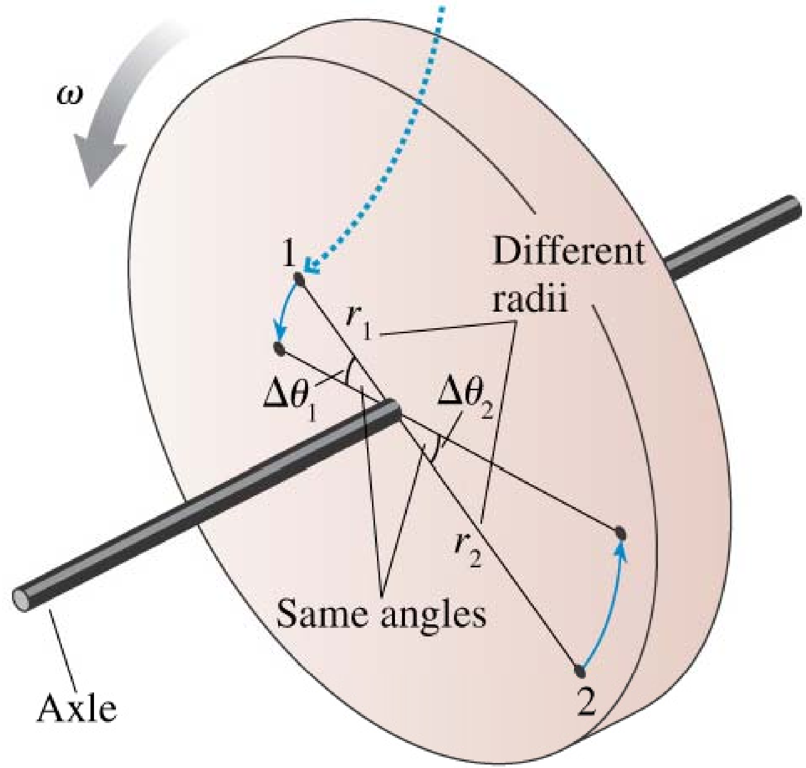
Linear acceleration:

$$a_1 = \frac{\Delta v_1}{\Delta t} = \frac{\Delta\omega}{\Delta t} r_1 \qquad a_2 = \frac{\Delta v_2}{\Delta t} = \frac{\Delta\omega}{\Delta t} r_2$$

Both points have the same angular acceleration $\alpha = \dfrac{\Delta\omega}{\Delta t}$

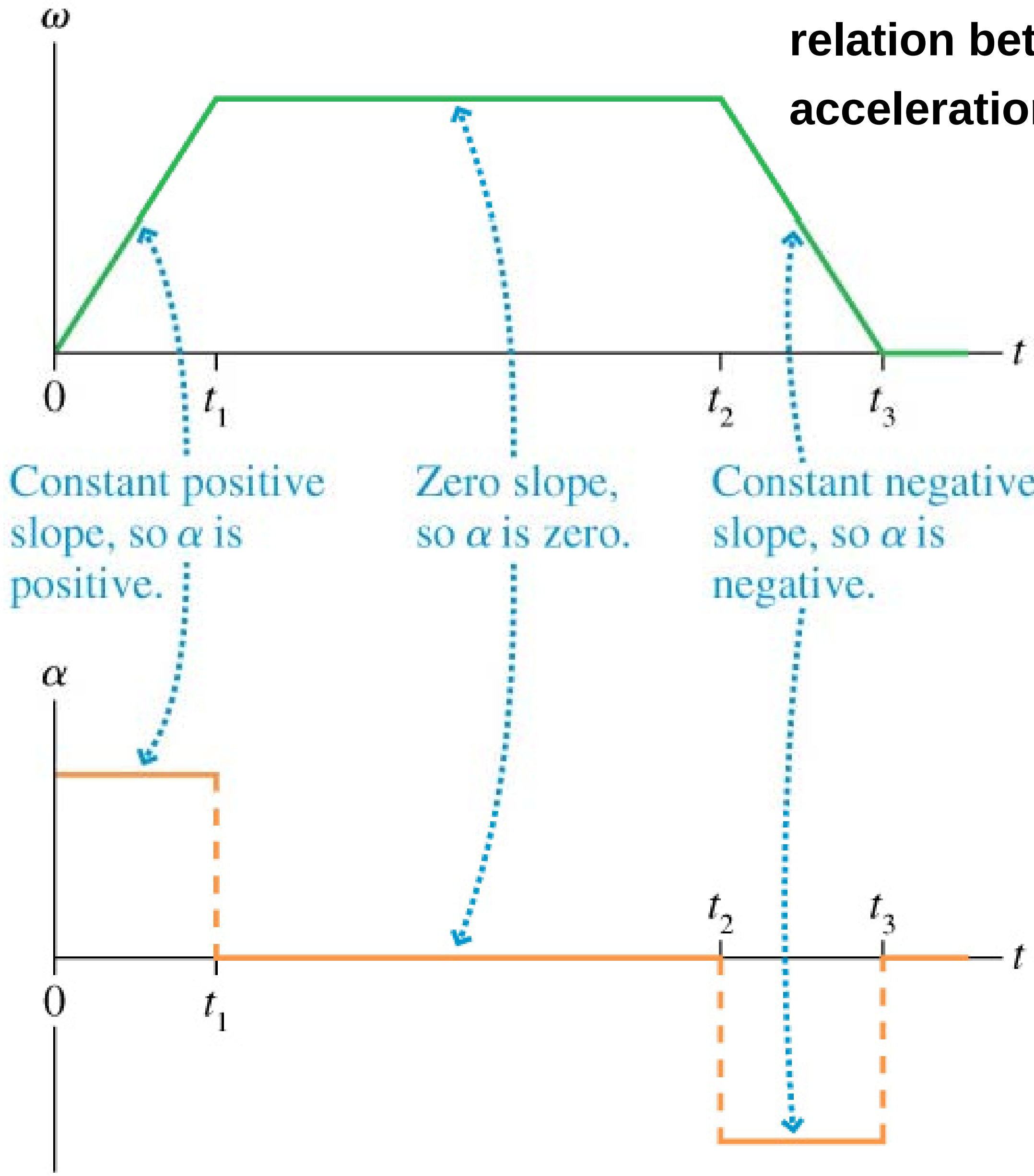
Rotation of Rigid Body:

Every point undergoes circular motion with the same angular velocity and the same angular acceleration



$$\alpha = \frac{\Delta\omega}{\Delta t} \qquad \boldsymbol{v = \omega r}$$

The relation between angular velocity and angular acceleration is the same as the relation between linear velocity and linear acceleration



$$\alpha = \frac{\Delta\omega}{\Delta t}$$

$$a = \frac{\Delta v}{\Delta t}$$

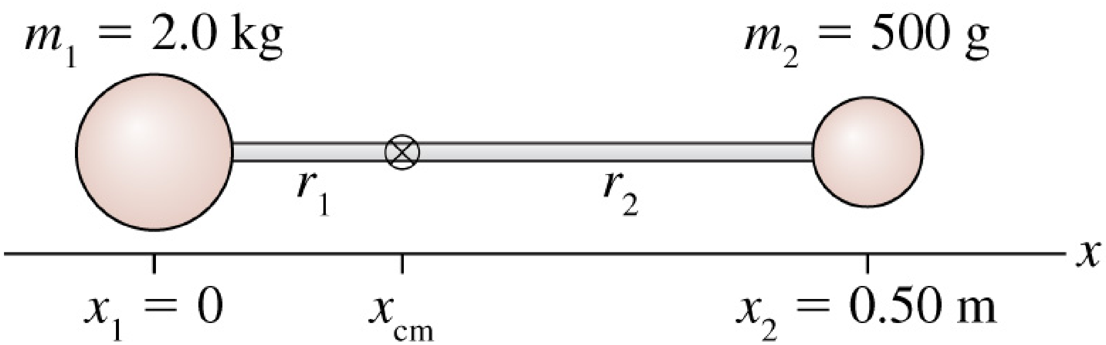
The Center of Mass

For Rigid Body sometimes it is convenient to describe the rotation about the special point– the center of mass of the body.

<u>Definition: The coordinate of the center of mass:</u>

Rigid body consisting of two particles:

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$



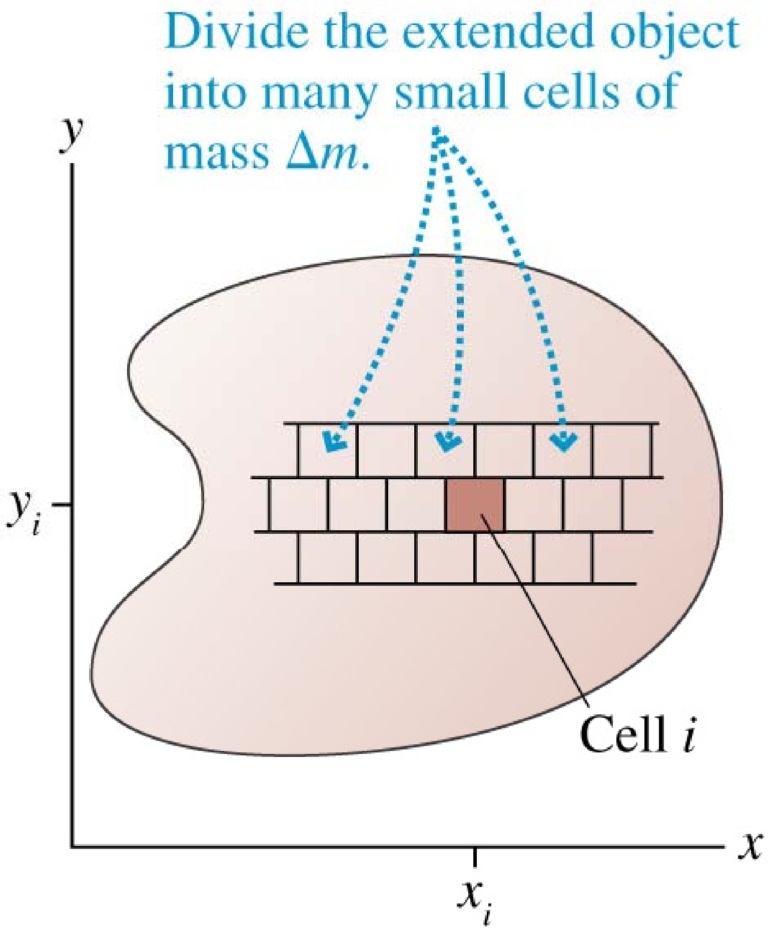
If $m_1 = m_2$ **then**

$$x_{cm} = \frac{x_1 + x_2}{2}$$

$$x_{cm} = \frac{2 \cdot 0 + 0.5 \cdot 0.5}{2.0 + 0.5} = 0.1m$$

The Center of Mass

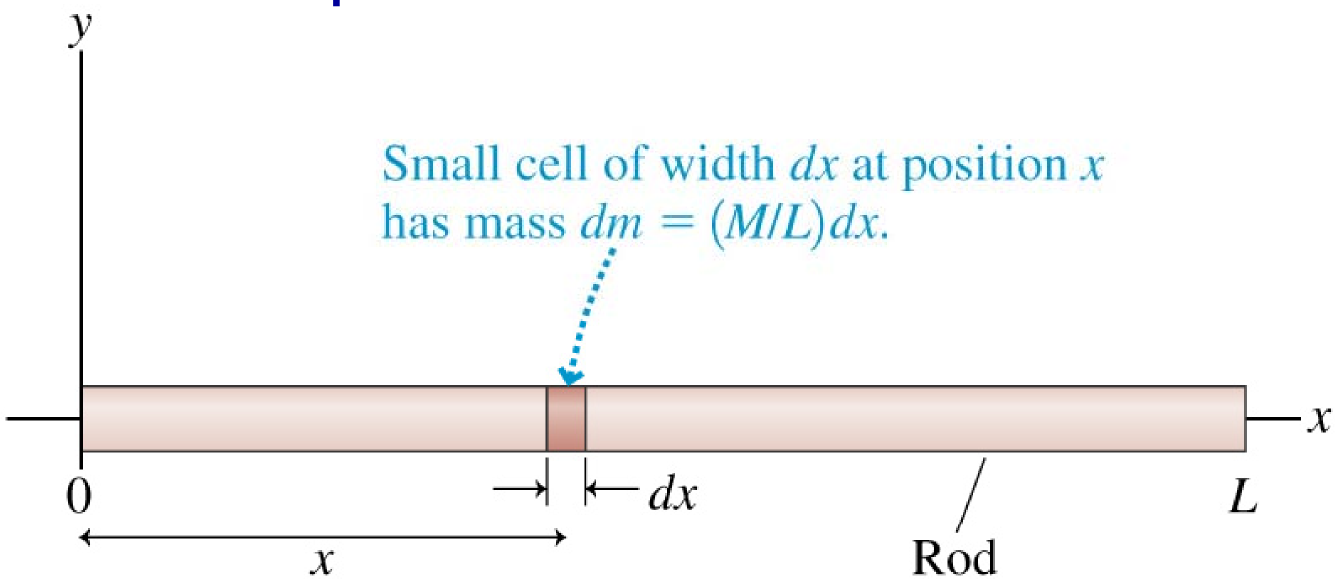
Definition: The coordinate of the center of mass:



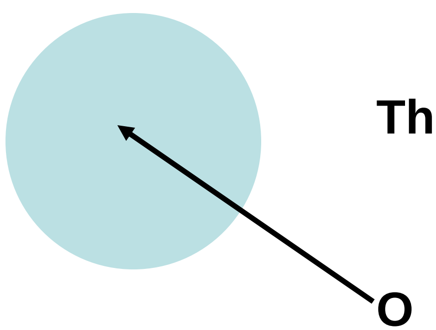
$$x_{cm} = \frac{\sum_i x_i \Delta m_i}{M} = \frac{\int x\,dm}{M}$$

$$y_{cm} = \frac{\sum_i y_i \Delta m_i}{M} = \frac{\int y\,dm}{M}$$

The Center of Mass: Example



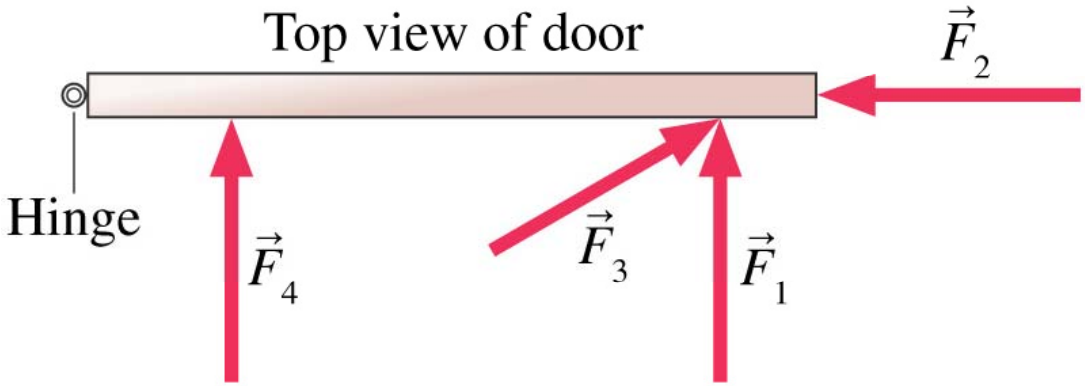
$$x_{cm} = \frac{\int x\,dm}{M} = \frac{\int_0^L x \frac{M}{L} dx}{M} = \frac{M \int_0^L x\,dx}{LM} = \frac{1}{L}\frac{1}{2} x^2 \Big|_0^L = \frac{L^2}{2L} = \frac{L}{2}$$



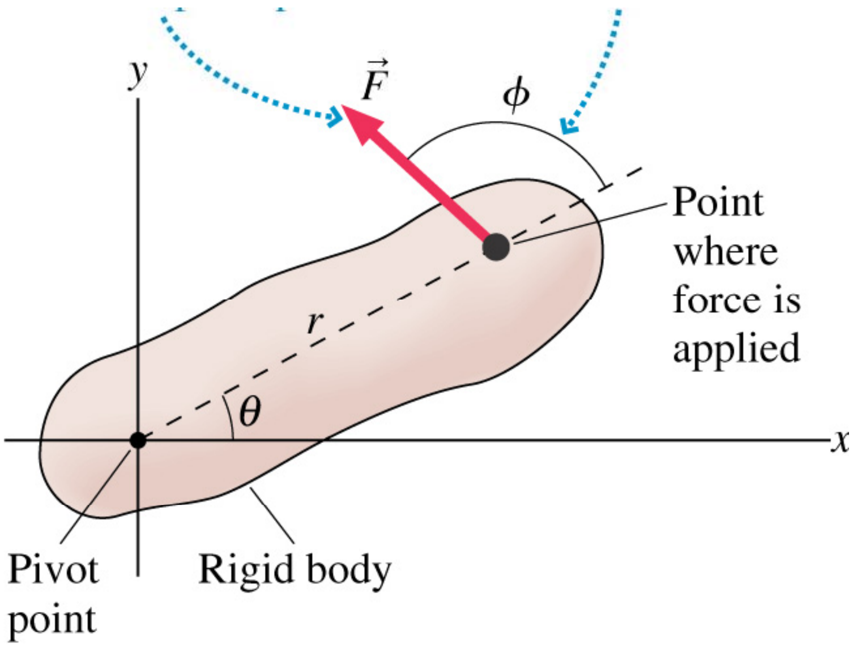
The center of mass of a disk is the center O of the disk

Torque: Rotational Equivalent of Force

Torque

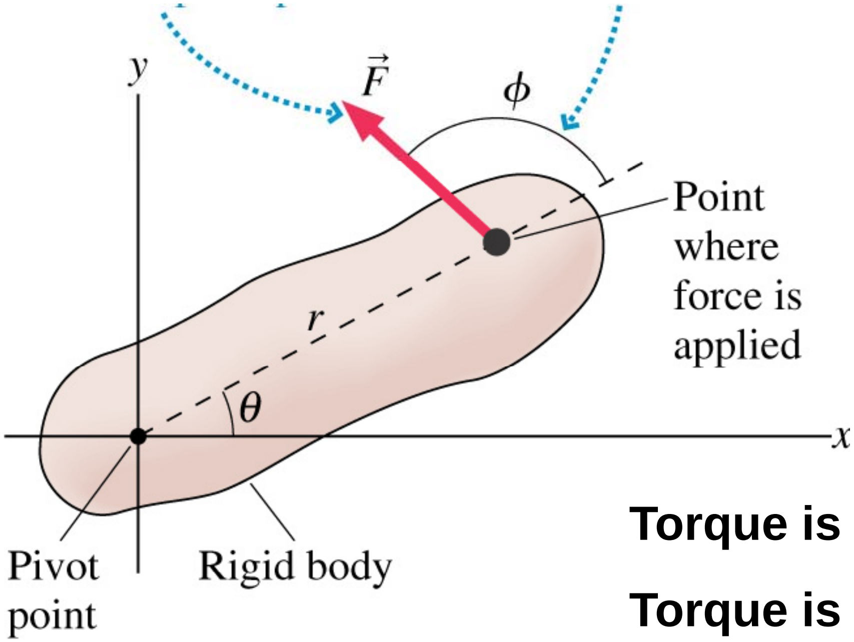


The rotation of the body is determined by the torque



$$\tau = Fr\sin\phi$$

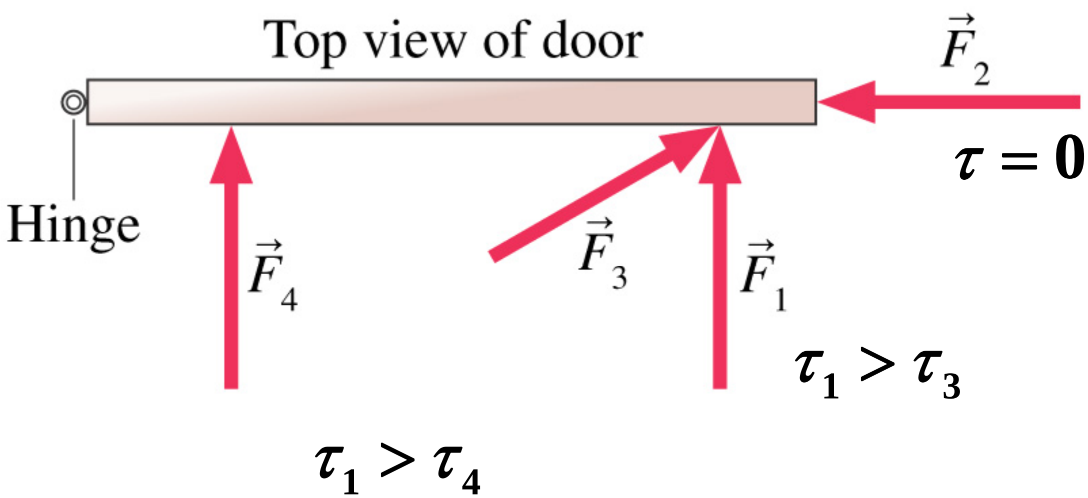
Torque



$$\tau = Fr\sin\phi$$

Torque is maximum if $\phi = 90^0$

Torque is 0 if $\phi = 0^0$ *or* $\phi = 180^0$

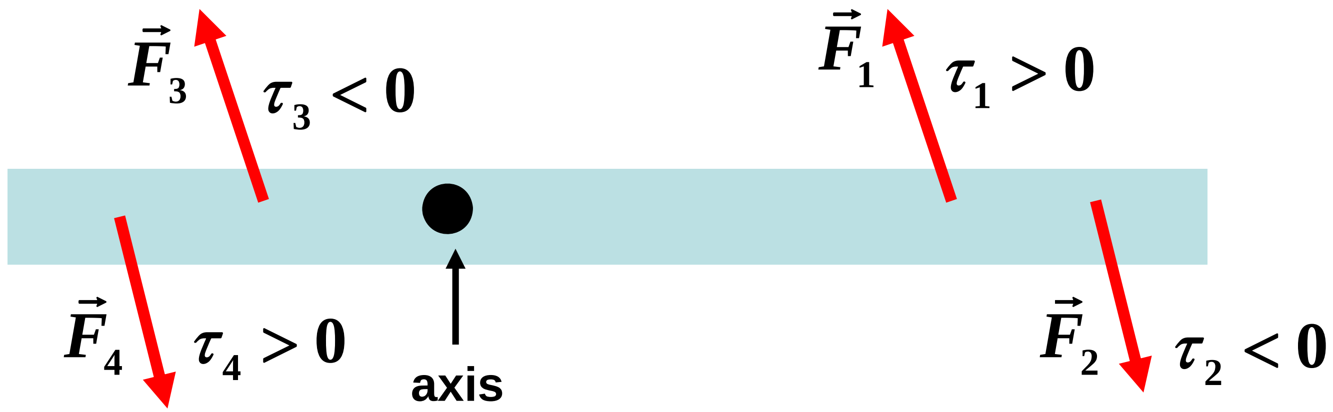


Torque

$$\tau = Fr\sin\phi$$

Torque is positive if the force is trying to rotate the body counterclockwise

Torque is negative if the force is trying to rotate the body clockwise



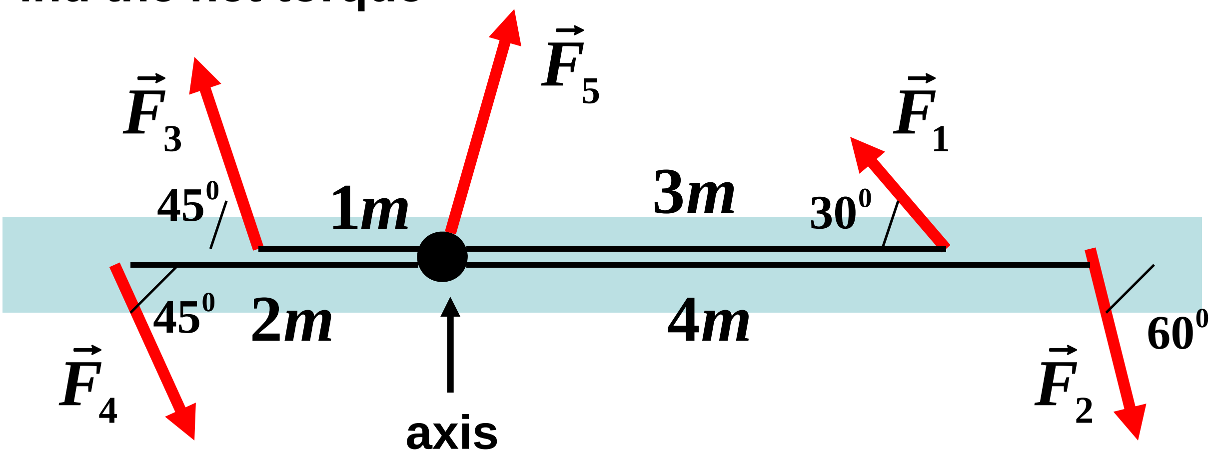
The net torque is the sum of the torques due to all applied forces:

$$\tau_{net} = \tau_1 + \tau_2 + \tau_3 + \tau_4$$

Torque: Example

$$\tau = Fr\sin\phi$$

Find the net torque



$$F_1 = F_2 = F_3 = F_4 = F_5 = 5N$$

$$\tau_1 = 5 \cdot 3\sin(30) = 7.5N \cdot m$$

$$\tau_2 = -5 \cdot 4\sin(60) = -17.3N \cdot m$$

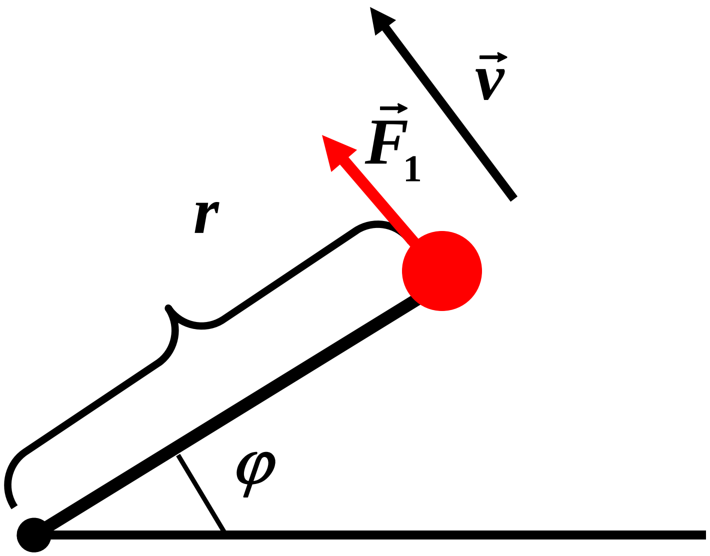
$$\tau_3 = -5 \cdot 1\sin(45) = -3.5N \cdot m$$

$$\tau_4 = 5 \cdot 2\sin(45) = 7.1N \cdot m$$

$$\tau_5 = 5 \cdot 0 = 0N \cdot m$$

$$\tau_{net} = \tau_1 + \tau_2 + \tau_3 + \tau_4 + \tau_4 = 7.5 - 17.3 - 3.5 + 7.1 = -6.2N \cdot m$$

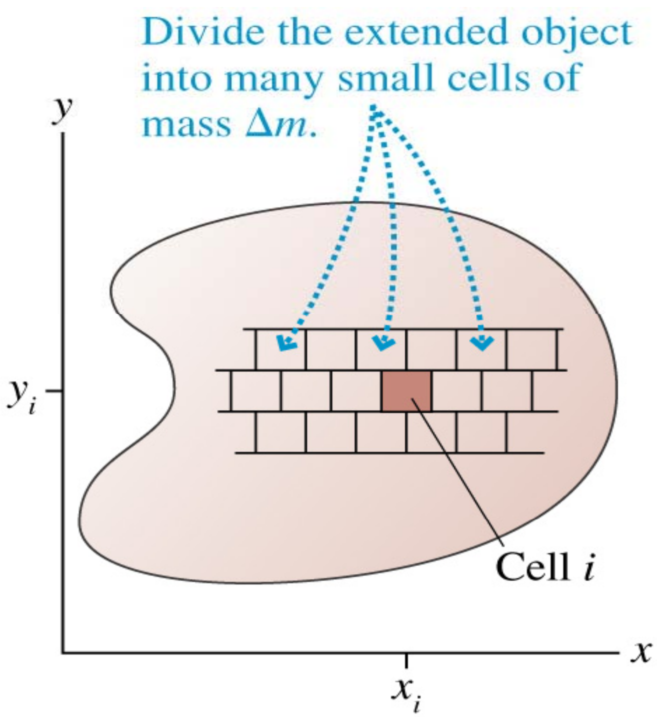
Torque: Relation between the torque and angular acceleration:



$$a = \frac{\Delta v}{\Delta t} = \frac{\Delta \omega}{\Delta t} r = \alpha r$$

$$F = ma = \alpha m r$$

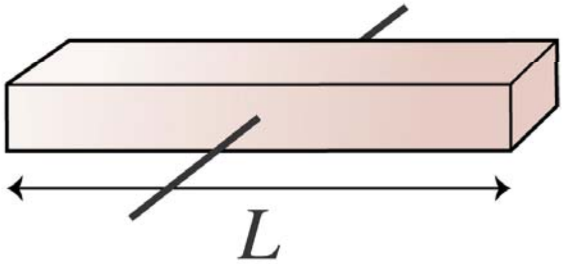
$$\tau = Fr = \alpha m r^2$$



$$\tau_{net} = \alpha \sum_i m_i r_i^2 = \alpha I$$

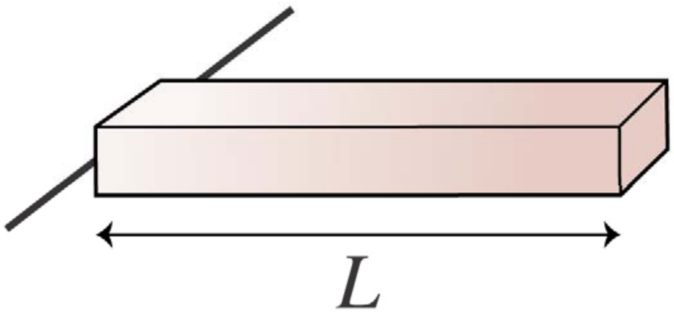
$$I = \sum_i m_i r_i^2$$ - moment of inertia

Moment of Inertia



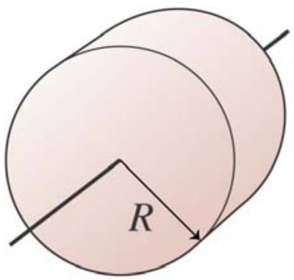
Thin rod, about center

$$I = \frac{1}{12} ML^2$$



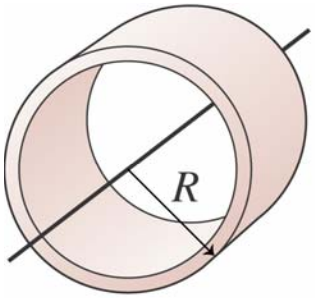
Thin rod, about end

$$I = \frac{1}{3} ML^2$$



Cylinder (or disk), about center

$$I = \frac{1}{2} MR^2$$



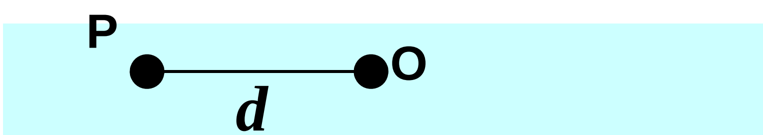
Cylindrical loop, about center

$$I = MR^2$$

Moment of Inertia: Parallel-axis Theorem

If you know the moment of Inertia about the center of mass (point O) then the Moment of Inertia about point (axis) P will be

$$I_P = I_O + Md^2$$



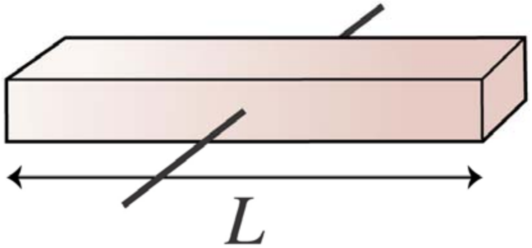
$$I_P = \sum_i (x_i - x_P)^2 \Delta m_i = \sum_i (x_i - x_O + x_O - x_P)^2 \Delta m_i =$$

$$= \sum_i (x_i - x_O + d)^2 \Delta m_i = \sum_i \left[(r_i - r_O)^2 + 2d(r_i - r_O) + d^2 \right] \Delta m_i =$$

$$+ \sum_i (r_i - r_O)^2 \Delta m_i + 2d \sum_i (r_i - r_O) \Delta m_i + Md^2 = I_O + Md^2$$

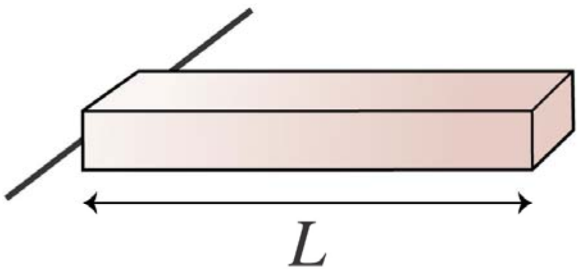
Parallel-axis Theorem: Example

$$I_P = I_O + Md^2$$



Thin rod, about center of mass

$$I = \frac{1}{12}ML^2$$

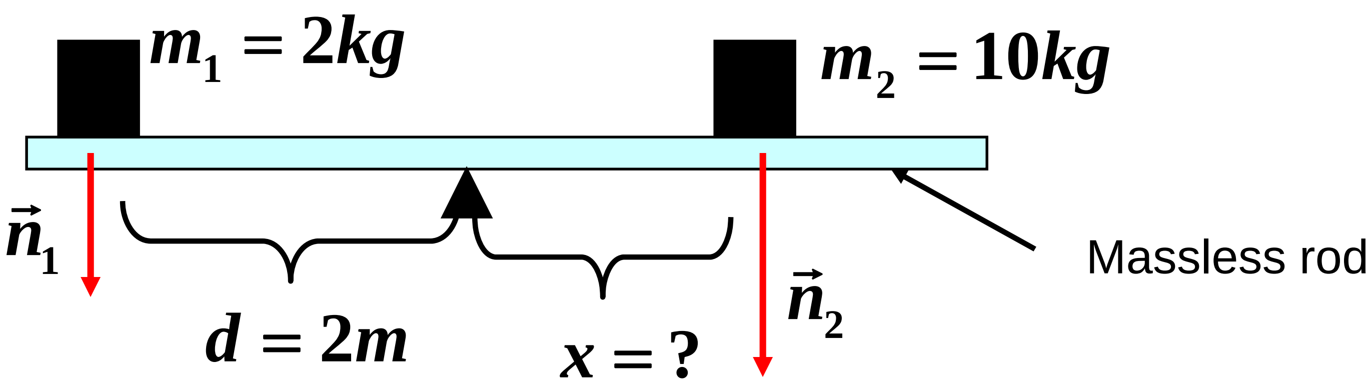


$$I = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{12}ML^2 + \frac{1}{4}ML^2 = \frac{1}{3}ML^2$$

$$\tau_{net} = \alpha I$$

Equilibrium: $\alpha = 0$

$$\tau_{net} = 0$$



Two forces (which can results in rotation) acting on the rod

$$n_1 = w_1 = m_1 g \qquad \tau_1 = n_1 d = m_1 g d$$

$$n_2 = w_2 = m_2 g \qquad \tau_2 = -n_2 x = -m_2 g x$$

Equilibrium: $\tau_{net} = \tau_1 + \tau_2 = 0$

$$m_1 g d - m_2 g x = 0$$

$$x = d\frac{m_1}{m_2} = 0.4m$$

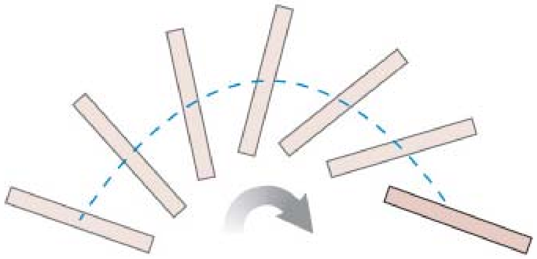
Rotational Energy:

$$K_{rot} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + ... =$$

$$= \frac{1}{2}m_1\omega^2 r_1^2 + \frac{1}{2}m_2\omega^2 r_2^2 + \frac{1}{2}m_3\omega^2 r_3^2 + ... =$$

$$= \frac{1}{2}\omega^2(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + ...) = \frac{1}{2}I\omega^2$$

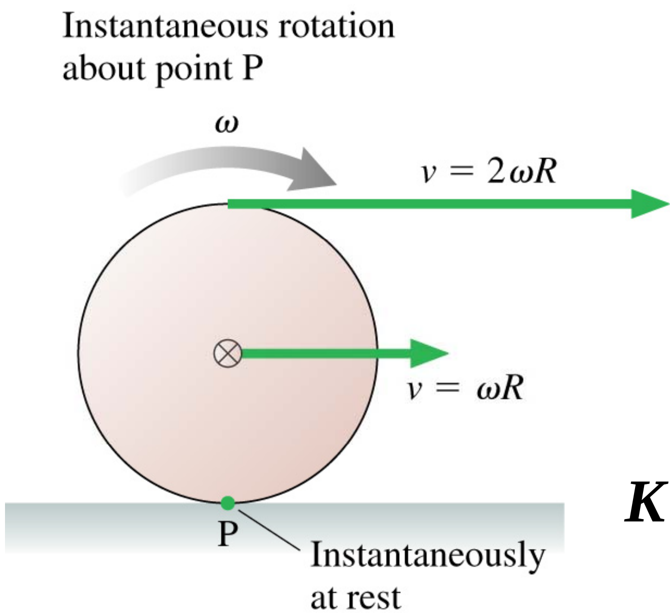
Conservation of energy (no friction):



$$Mgy + \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = const$$

Kinetic energy of rolling motion

$$K_{rot} = \frac{1}{2} I\omega^2$$



$$K_{rot} = \frac{1}{2} I_P \omega^2 \qquad I_P = I_{cm} + MR^2$$

$$v_{cm} = \omega R$$

$$K_{rot} = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} MR^2\omega^2 = \frac{1}{2R^2} I_{cm} v_{cm}^2 + \frac{1}{2} Mv_{cm}^2$$

Cylinder: $I_{cm} = \frac{1}{2} MR^2$ $\quad K_{rot} = \frac{1}{2}\frac{1}{2} Mv_{cm}^2 + \frac{1}{2} Mv_{cm}^2 = \frac{3}{4} Mv_{cm}^2$

Cylindrical loop: $I_{cm} = MR^2$ $\quad K_{rot} = \frac{1}{2} Mv_{cm}^2 + \frac{1}{2} Mv_{cm}^2 = Mv_{cm}^2$

Solid sphere: $I_{cm} = \frac{2}{5} MR^2$ $\quad K_{rot} = \frac{1}{2}\frac{2}{5} Mv_{cm}^2 + \frac{1}{2} Mv_{cm}^2 = \frac{7}{10} Mv_{cm}^2$